1. The following picture shows a whole strip with portions shaded, starting at the left.

First I shaded one third of the strip (the left-most third). Then I shaded one ninth of the original strip (adjacent to the part I just shaded), then one twenty-seventh of the original strip (adjacent to the part I just shaded), as so on, with each portion that I shade being one third of the size of the portion to its left. Suppose I carry out this process forever. We want to know what fraction of the strip will end up shaded.

(a) (4 points) Looking at the picture, estimate the fraction of the strip you think will end up shaded. Explain why you think so.

(b) (6 points) Use what you have learned about infinite geometric sums to find the exact answer.
2. Consider the infinite sum

\[
\frac{1}{0.5} + \frac{1}{1} + \frac{1}{1.5} + \frac{1}{2} + \frac{1}{2.5} + \frac{1}{3} + \frac{1}{3.5} + \cdots.
\]

The terms are getting smaller and smaller. For example, the two-millionth term is 0.000001. I asked a computer to calculate the sum of the first two million terms, and it told me the answer is approximately 30.171473.

(a) (3 points) Compute the sum of the first three terms by hand (get an exact answer, expressed as a fraction or mixed number).

(b) (3 points) Find the sum of the first six terms using a calculator to get a decimal approximation.

(c) (7 points) Discuss what the sum of all the terms of this series is. Explain your reasoning. [HINT: We talked about a series like this in class.]
3. Imagine that each point on the real number line (i.e., each real number) can be colored either black or gold. Let $P$ be the set of all ways to color the entire line. For example, one coloring is to have all the numbers less than 17 be black and all the numbers 17 or greater be gold. Another coloring would be to let all the numbers that can be written as finite decimals (like 3.14) be black and all the other real numbers (like $\pi$ and $1/3$) be gold. This problem explores how big $P$ is.

(a) (6 points) Explain why the cardinality of $P$ is at least $2^{\aleph_0}$, which as we know is the cardinality of the set of all real numbers. [HINT: Find $2^{\aleph_0}$ different colorings. For example, think about the colorings like the first example mentioned above.]

(b) (8 points) Prove that the cardinality of $P$ is greater than $2^{\aleph_0}$. [HINT: Come up with a Cantor-like diagonal argument. Assume that there is a pairing of all the real numbers with all the colorings. Construct a new coloring by cleverly assigning a color to each number $x$ based on the coloring that is paired with $x$. Show that this new coloring is not paired with any real number. Be verbose!]
4. Recall that we made an ordered transfinite list of all ordinals, where each ordinal in the list was the set consisting of all the previous ordinals. Recall that we began with the empty set (the set with no elements), which we named 0, and we used the names 1, 2, and so on for the finite ordinals, and then used the name $\omega$ for the next ordinal after all of those (so $\omega = \{0, 1, 2, \ldots\}$). Thus our list began $0, 1, 2, 3, \ldots, \omega$.

(a) (3 points) What came next? In other words, fill in the blank:

$$0, 1, 2, 3, \ldots, \omega, \_ \_ \_$$

(b) (3 points) What came immediately after infinitely many terms after $\omega$? In other words, fill in the blank:

$$0, 1, 2, 3, \ldots, \omega, [\text{your answer to (a)}], \ldots, \_ \_ \_$$

(c) (5 points) Eventually our list had an ordinal we called $\omega_1$. Explain what $\omega_1$ is.

5. (7 points) Sally and Bob are playing the children’s game of trying to outdo each other by naming bigger and bigger cardinal numbers. But Sally and Bob are pretty sophisticated children, so they typically deal with infinite cardinal numbers, not those mundane finite ones like the number of dollars the United States government owes it creditors (currently about 8.85 trillion). If Sally names a set $A$ and says that her number is the size of $A$ (which we write $n(A)$), how can Bob beat her?
6. Consider the number \( x = 12.152525252\ldots \) (This can also be written as 12.1\overline{52}).

(a) (5 points) Explain completely, using words, math symbols, and/or diagrams, what all those ink marks on paper really mean. How does one make sense out of this infinite decimal? How do we get a handle on what real number we are talking about here?

(b) (6 points) If this number \( x \) is rational, express it as \( a/b \) for natural numbers \( a \) and \( b \) (you need to show how you computed this, not just use a calculator button). If it is irrational, explain clearly why it cannot be expressed in this way.
7. The two infinite cardinalities we talked about most were $\aleph_0$ and $2^{\aleph_0}$ (also called $c$).

(a) (4 points) Name two very different infinite countable sets (i.e., sets that have cardinality $\aleph_0$).

(b) (5 points) Show that the two sets you named in part (a) have the same cardinality.

(c) (5 points) Name two very different sets that both have cardinality $c$. (No justification is needed for the answer to this question.)
8. (5 points) Precisely state the Schröder–Bernstein Theorem. (Recall that this theorem, which we proved by looking at a diagram with lots of arrows, enables one to show that two sets have the same cardinality without explicitly exhibiting a one-to-one correspondence between their elements.)

The remaining five problems are multiple choice questions based on the reading in Moore’s book and our discussions of that material. In each case choose the one best answer by circling the letter of your choice. Each question counts 5 points for the correct response, 0 points for an incorrect response (so guess any you don’t know).

9. Who was it that proposed a “programme” asking mathematicians to prove the consistency of their axioms using finitary methods, so that we would know we were on sound footing when proving statements about infinite sets?
   (a) Hegel  
   (b) Hilbert  
   (c) Kant  
   (d) Russell  
   (e) Wittgenstein

10. Who was it that found a paradox in set theory arising from allowing something like the set of all sets to exist?
   (a) Hegel  
   (b) Hilbert  
   (c) Kant  
   (d) Russell  
   (e) Wittgenstein
11. Who was it that described antimonies (paradoxes) about the infinitely big and the infinitely small and ended up concluding that such questions were meaningless as applied to the physical world?

(a) Hegel  
(b) Hilbert  
(c) Kant  
(d) Russell  
(e) Wittgenstein

12. Who was Georg Cantor?

(a) A nineteenth century German mathematician who did ground-breaking research on infinite cardinal numbers.  
(b) A twentieth century Austrian logician who showed that the continuum hypothesis could not be disproved from accepted axioms of set theory.  
(c) An eighteenth century German philosopher who proved that infinite sets did not exist.  
(d) An nineteenth century existentialist philosopher from Germany who is best known for his statement that “God is dead.”  
(e) A mathematics professor currently at Stanford University who showed that the continuum hypothesis could not be proved from accepted axioms of set theory.

13. Which of these best expresses the point of view taken by the intuitionists (such as Brouwer) about proving mathematical theorems?

(a) Mathematical statements are about finite objects, and proofs have to be constructive.  
(b) Anything goes—as long as we have some intuition about infinite sets, we can invoke that intuition in our proofs.  
(c) Nothing can be proved in mathematics.  
(d) Diagonal arguments show that infinite sets can have different sizes.  
(e) For any conjecture about numbers, such as Goldbach’s conjecture, either the conjecture is true or there is a counterexample.