1. A copper cooking pot needs to be coated with a thin layer of tin (or some other material) on the inside, since it is not desirable to have copper come into contact with the food being cooked in the pot. The tin gradually wears away, and a new layer of tin needs to be applied every so often. I recently sent an open pot 10" in diameter and 4" deep to a firm in New York to be retinned. It came back with a shiny new coat of tin on the bottom and lateral surface of the inside of the pot (the outside is not tinned, of course). Their formula for the cost of retinning is as follows: take the diameter, add twice the height, and multiply the result by $1.25.

(a) How much did they charge me for retinning?
(b) What would the charge have been if the pot were only half as big—i.e., half as wide and half as tall? Obtain this answer without doing a new calculation from scratch, but explain why your answer is correct.
(c) One would assume that the cost of retinning should depend on (and be proportional to) the area of the surface to be retinned. Compute the area of my pot that was retinned, and compute the charge per square inch for retinning.
(d) What would the area and charge per square inch have been if the pot were only half as big—i.e., half as wide and half as tall? Obtain this answer without doing a new calculation from scratch, but explain why your answer is correct.
(e) Based on what you have found in this problem, you should conclude that their pricing policy doesn’t make much sense. Explain why, and then come up with a good explanation for why they use it anyway.

2. Recall that a regular octahedron consists of eight equilateral triangles joined in such a way that at each vertex there are four triangles. One way to think of it is as two congruent right square pyramids sharing a common base, pointing out in opposite directions. It can be shown that this polyhedron can be inscribed in a sphere.

(a) Draw a neat picture of the regular octahedron, and then sketch a sphere circumscribing it. [You may want to sketch this on some scratch paper first—see also part (b).]
(b) Determine the fraction of the volume of the sphere that the octahedron occupies. Give your answer to the nearest percent. You may wish to recall that the volume of a sphere is $4/3\pi r^3$. [Hint: the center of the sphere is the middle of the square common base of the two pyramids that make up the octahedron. Draw some useful radii of the sphere; call the radius $r$. Compute the length of an edge of the octahedron (they’re all equal!) using the Pythagorean Theorem.

3. Give a ruler and compass construction for the following: given two intersecting lines $l$ and $m$, find a circle tangent to both $l$ and $m$. Write out what your construction is in words (numbered steps would be a good format), and show it clearly with a drawing. [Hint: there are an infinite number of such circles; you will need to locate the center of your circle, of course.]

4. Fill in the blanks with the correct technical term from this course. Spelling counts.

(a) Two figures $A$ and $B$ in the plane are ____________________ if and only if there is a composite of reflections, translations, and/or rotations that sends $A$ into $B$.
(b) Two lines are ____________________ if and only if they intersect and form a right angle.
(c) Two lines are ____________________ if and only if they do not intersect but do not lie in the same plane.
(d) A ________________ is a 4-sided polygon in which each pair of opposite sides is parallel.

(e) The ________________ of angle $\angle ABC$ is the set of points in the plane that are in the same halfplane determined by line $\overrightarrow{AB}$ as $C$ and in the same halfplane determined by line $\overrightarrow{BC}$ as $A$.

(f) A ________________ is the set of all points in space at an equal distance from a fixed point.

(g) A ________________ is the unit of measure equal to one thousandth of a meter.

5. Show that the sum of the measures of the angles in a triangle is 180 degrees. Explain clearly and fully.
   [Hint: draw a line through one vertex of the triangle, parallel to the opposite side; then discuss alternate interior angles.]

6. Solve the following problem suggested by one of the activities in the NCTM Standards: What is the smallest number of toothpicks you need in order to form a scalene triangle using the toothpicks, laid end to end, to make up the sides of the triangle? Justify your answer by drawing the picture for the number of toothpicks you say are required, and explaining why fewer toothpicks will not work.

7. Which is going faster, a car traveling at 40 kilometers per hour or a race horse running a mile in 2 minutes? Justify your answer.

8. [NOTE FROM 1996: This problem made more sense to the students in 1990, because they had a project dealing with pentominoes.] A hexomino is like a pentomino except that it is made up of six squares joined along their edges instead of five. It turns out that there are 35 different hexominoes. Draw a hexomino having each of the following symmetries:
   (a) Reflective symmetry but no rotational symmetry.
   (b) Rotational symmetry but no reflective symmetry.
   (c) Neither reflective nor rotational symmetry.

9. Two quick questions about measurement:
   (a) Suppose that a solid bronze replica of a famous Rodin sculpture of a man is 10 inches high and weighs 9 pounds. How much would you expect a replica of the same sculpture, made out of the same material, but 20 inches tall, to weigh?
   (b) Pamela measures the side of a square piece of paper and records its length as 40.1 centimeters. What, if anything, is wrong with the following statement? “The area of the piece of paper is 1608.01 cm$^2$.”

10. John, Bob, and Alice are sitting in a triangle. John notices that the angle formed by the rays from him to the other two has measure $37^\circ$, and Bob notices that the angle formed by the rays from him to the other two also has measure $37^\circ$.
    (a) Is John closer to Alice or to Bob?
    (b) Is Alice closer to John or to Bob?

11. In the figure shown here, consider the transformation that consists of a reflection through line $m$, followed by a rotation of $90^\circ$ clockwise around point $P$.

![Diagram](image-url)
(a) Show the position of the letter A shape under this transformation. (Draw it on the figure above, making sure I can tell what you intend your answer to be.)

(b) Find a transformation or sequence of transformations that undoes what this transformation did. In other words, what transformation will send the A you drew back to the A in the original figure?

12. Locate three points, A, B, and P, in the plane, not collinear. Then using straightedge and compass only, construct the image of P under the translation that takes A to B. Make sure it is clear what you are doing (preferably, write down your steps).

13. A right triangle has one side of length 7 and the hypotenuse of length 25. Find its area.

14. Give clear and precise definitions of the following: (a) an angle; (b) hexagon ABCDEF is congruent to hexagon UVWXYZ; (c) the interior of a circle; (d) the image of point P under the rotation of the plane through a 35° angle, clockwise around point C [you probably want to draw a picture to go along with your verbal description].

15. Draw examples of geometric figures with the following properties:
   (a) a figure with 90° rotational symmetry but no reflectional symmetry
   (b) a figure with reflectional symmetry but no rotational symmetry

16. Determine the number of diagonals that a convex centagon (100-sided polygon) has. You need to give some verbal justification for your answer—just plugging into a formula you have memorized will not receive full credit.

17. From any point P outside a circle C there are exactly two tangent lines to the circle. Draw a picture of this situation. Are the lengths of these two tangent segments from P to C equal? Give an argument to support your answer.

18. A swimming pool is 120 feet long and 50 feet wide, and its depth varies evenly from 3 feet at the shallow end to 11 feet at the deep end. Find the approximate amount of water in the pool in gallons, if a gallon occupies about 231 cubic inches. State the answer with an appropriate amount of precision. [Hint: the figure of interest here is a trapezoidal prism.]

19. Consider a right regular pentagonal pyramid (the base is a regular 5-sided polygon).
   (a) Draw a picture of this object.
   (b) Verify that Euler’s formula \( V - E + F = 2 \) holds for this figure, by counting the number of vertices, the number of edges, and the number of faces, and doing the arithmetic.
   (c) Suppose that the area of the base of this pyramid happens to be 2.3 cm\(^2\) and the height happens to be 3.7 cm. Find the volume of the pyramid (state it in the correct units, with an appropriate amount of precision).
   (d) Suppose also that the length of each side of the pentagon is 1.0 cm and that the slant height of the pyramid is 3.8 cm. Find the total surface area of the pyramid (state your answer in the correct units, with an appropriate amount of precision).

20. A ball fits snugly inside a closed box. (Or, to put it mathematically, a sphere is inscribed in a cube.)
   (a) What fraction of the space inside the box is taken up by (the inside of) the ball? Express your answer to the nearest whole percent. You may wish to recall that the formula for the volume of a sphere is \( V = \frac{4}{3}\pi r^3 \).
(b) How do the total surface area of the ball and the box compare? Specifically, what fraction of the area of the box is the area of the ball? Express your answer to the nearest whole percent. You may wish to recall that the formula for the area of a sphere is \( A = 4\pi r^2 \).

Here are some multiple choice problems.

21. What is the total surface area of a cylinder?
   (a) \( 2\pi r(r + h) \)  (b) \( \pi rh \)  (c) \( 2\pi rh \)  (d) \( \frac{\pi rh}{3} + 2\pi r^2 \)

22. In the figure shown here, what is the relationship between the measures of the angles \( \angle AOB \) and \( \angle ACB \)?

![Diagram of a circle with points A, B, and C, and a line segment from O to AB]

(a) Their sum is \( 180^\circ \).
(b) Their difference is \( 90^\circ \).
(c) \( m\angle AOB \) is twice as much as \( m\angle ACB \).
(d) \( m\angle ACB \) is twice as much as \( m\angle AOB \).

23. If \( X, Y, \) and \( Z \) are three distinct points, under what conditions will there be a circle containing all three of them?
   (a) always
   (b) if and only if they are not collinear
   (c) if and only if triangle \( \triangle XYZ \) is acute
   (d) if and only if triangle \( \triangle XYZ \) is equilateral

24. If a circle has circumference \( x \), then what is its area?
   (a) \( \pi x^2 \)  (b) \( \pi \left(\frac{x}{2}\right)^2 \)  (c) \( \frac{x^2}{4\pi} \)  (d) \( \frac{x}{2} \)

25. Which of these figures does not tessellate (tile) the plane?
   (a) any convex quadrilateral
   (b) a regular hexagon
   (c) a regular pentagon
   (d) a right isosceles triangle
26. Which of these is closest to being 100 mm?
   (a) the width of your hand
   (b) the circumference of your head
   (c) your height
   (d) the length of your foot

27. Which of the following statements is true about regular polygons and regular polyhedra? By “essentially different” here I mean having different numbers of edges, sides, or faces.
   (a) There are an infinite number of essentially different regular polygons and an infinite number of essentially different regular polyhedra.
   (b) There are an infinite number of essentially different regular polygons but only a finite number of essentially different regular polyhedra.
   (c) There are an infinite number of essentially different regular polyhedra but only a finite number of essentially different regular polygons.
   (d) There are only a finite number of essentially different regular polygons and only a finite number of essentially different regular polyhedra.

28. What is the area of the interior of a polygon constructed on a geoboard such that it has 17 nails in the interior and 24 nails around the boundary?
   (a) 28   (b) 29   (c) 41   (d) 42

29. Which of these is the best physical example of a simple closed surface?
   (a) a basketball
   (b) a sheet of paper
   (c) the surface of a donut
   (d) the space inside a shoe box

30. Which of these do not determine a unique plane containing them?
   (a) a line and a point not on the line
   (b) two skew lines
   (c) three noncollinear points
   (d) two intersecting lines

31. Square pyramid $A$ has a volume of $20 \text{ cm}^3$. Square pyramid $B$ has a base that is twice as long, and pyramid $B$ is three times as high, as pyramid $A$. What is the volume of pyramid $B$?
   (a) $100 \text{ cm}^3$   (b) $120 \text{ cm}^3$   (c) $160 \text{ cm}^3$   (d) $240 \text{ cm}^3$

32. Suppose that $\triangle ABC$ and $\triangle DEF$ are two triangles. Which of the following sets of facts is not sufficient to imply that $\triangle ABC$ and $\triangle DEF$ are congruent?
   (a) $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, and the triangles have the same area
   (b) $AB = DE$, $AC = DF$, and $BC = EF$
   (c) $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $BC = EF$
   (d) $AB = DE$, $BC = EF$, and $\angle A \cong \angle D$