One of computer scientists’ favorite functions is the Ackermann function, first studied by David Hilbert and Wilhelm Ackermann about 75 years ago [GZ]. It is recursive (i.e., computable), but it grows too fast to be primitive recursive (i.e., computable without using dirty tricks like double recursion or the operator “the least $n$ such that”). A kind of inverse for this function (which grows excruciatingly slowly—it makes something like log(log($n$)) look like the U.S. national debt by comparison) enters into the efficiency analysis of some important algorithms, such as keeping track of the components of a graph as new edges are added.

If we restrict the range of the Ackermann function to a finite set (with a suitable “mod”-ification of its definition), then we might expect the exuberance of the original function to be reflected in rather chaotic behavior within this set. In fact we seem to find just the opposite, with the finitized Ackermann function petering out very quickly. We have many partial results about this mod-$n$ Ackermann function, obtained using fairly straightforward ad hoc arguments as well as a little elementary number theory. We also have some intriguing experimental data. Perhaps some readers of this article can provide a more definitive description of what’s going on.

To be specific, let $\mathbb{N}$ denote the set $\{0, 1, 2, 3, \ldots\}$ of natural numbers, and for each integer $n > 2$ let $\mathbb{N}_n$ denote the set $\{0, 1, 2, \ldots, n - 1\}$ of natural numbers less than $n$. Define the standard mod-$n$ Ackermann function from $\mathbb{N} \times \mathbb{N}_n$ to $\mathbb{N}_n$ by

$$A_n(i, j) = \begin{cases} 
(j + 1) \text{ mod } n & \text{if } i = 0 \\
A_n(i - 1, 1) & \text{if } i > 0 \text{ and } j = 0 \\
A_n(i - 1, A_n(i, j - 1)) & \text{if } i > 0 \text{ and } j > 0.
\end{cases}$$
\begin{figure}
\centering
\begin{tabular}{cccccccccc}
12 & 0 & 1 & 1 & 5 & 3 & 5 & 9 & 9 & \ldots \\
11 & 12 & 0 & 12 & 1 & 2 & 9 & 9 & 9 & \ldots \\
10 & 11 & 12 & 10 & 12 & 6 & 5 & 9 & 9 & \ldots \\
9  & 10 & 11 & 8 & 11 & 5 & 9 & 9 & 9 & \ldots \\
8  & 9  & 10 & 6 & 4  & 0 & 5 & 9 & 9 & \ldots \\
7  & 8  & 9 & 4 & 7 & 1 & 9 & 9 & 9 & \ldots \\
6  & 7 & 8 & 2 & 2 & 11 & 5 & 9 & 9 & \ldots \\
5  & 6 & 7 & 0 & 6 & 9 & 9 & 9 & 9 & \ldots \\
4  & 5 & 6 & 11 & 8 & 3 & 5 & 9 & 9 & \ldots \\
3  & 4 & 5 & 9 & 9 & 2 & 9 & 9 & 9 & \ldots \\
2  & 3 & 4 & 7 & 3 & 6 & 5 & 9 & 9 & \ldots \\
1  & 2 & 3 & 5 & 0 & 5 & 9 & 9 & 9 & \ldots \\
\end{tabular}
\caption{The standard mod-13 Ackermann function $A_{13}$.}
\end{figure}

A (possibly) nonstandard mod-$n$ Ackermann function is defined in the same way, except that the values $A_n(0,j)$ for $j = 0, 1, 2, \ldots, n - 1$ are arbitrary. We will write $A_n^\ast$ to refer specifically to the standard function. The value $A_n(i,j)$ is said to be in the $i$th column and $j$th row; we picture these values arranged as in Figures 1 and 2.

\begin{figure}
\centering
\begin{tabular}{cccccccccc}
6  & 6 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
5  & 0 & 4 & 0 & 4 & 0 & 4 & 0 & 0 & \ldots \\
4  & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
3  & 5 & 5 & 0 & 4 & 0 & 0 & 0 & 0 & \ldots \\
2  & 6 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1  & 0 & 4 & 0 & 4 & 0 & 4 & 0 & 0 & \ldots \\
\end{tabular}
\caption{A nonstandard mod-7 Ackermann function $A_7$.}
\end{figure}

If we set $n = \infty$, then we obtain in the standard case one of the usual versions of the nonfinitized Ackermann function. It grows monotonically (and wildly) as $i$ and $j$ increase; for example, $A_\infty^\ast(2,3) = 9$, $A_\infty^\ast(3,3) = 61$, and $A_\infty^\ast(4,3)$ has about $10^{20000}$ digits.

Let us adopt the following terminology. Denote the set of values that appear in the
ith column by $P_n(i)$. Clearly $P_n(0) \supseteq P_n(1) \supseteq P_n(2) \supseteq \cdots$; denote the intersection of this sequence, $\bigcap_{i=0}^{\infty} P_n(i)$, by $P_n$. If $A_n$ becomes constant in some column $i$, i.e., $A_n(i,0) = A_n(i,1) = \cdots = A_n(i,n-1)$, then the function is said to have stabilized in column $i$ (and clearly remains constant in all subsequent columns). The smallest $i$, if any, such that $A_n$ has stabilized in column $i$ is called the stability number of $A_n$, denoted $s(n)$ in the case of the standard mod-$n$ Ackermann function.

For $n < \infty$ only two kinds of asymptotic behavior are possible (since there are only finitely many different columns, and each column is uniquely determined by the one before it): either $A_n$ stabilizes, or the columns are (nontrivially) periodic, i.e., for some $t > 1$, $A_n(i,j) = A_n(i+t,j)$ for all $j$ and large enough $i$. In the nonstable case the smallest $t$ for which this occurs is called the period.

Figures 1 and 2 illustrate the only two known ways in which any mod-$n$ Ackermann function behaves asymptotically. In Figure 1 we see that $s(13) = 6$. This behavior, in which the function stabilizes fairly quickly, seems to happen in almost all cases, standard or not. On the other hand, in Figure 2, we see a nonstable situation for a nonstandard mod-$n$ Ackermann function, in which the period is 2. It is easy to construct an example of this type for any even positive integer $m$, i.e., a nonstandard mod-$n$ Ackermann function with period 2, whose columns eventually alternate between $(m,0,0,\ldots)$ and $(0,m,0,\ldots)$, in fact starting with a permutation of $\mathbb{N}_n$ in column 0 as long as $m > 2$.

Here is what we have found computationally. The only value of $n < 1,000,000$ for which the standard mod-$n$ Ackermann function does not stabilize is $n = 1969$. (The first author’s older child has been searching for some mystical significance to this property of his birth year.) For $n = 1969$ the period 2 behavior starts in column 8, with the columns alternating between $(1698,0,0,\ldots)$ and $(0,1698,0,1698,\ldots)$. For all other $n < 500,000$, the stability number for the standard function is at most 15, and is usually much less (for example, it often happens that $s(n) = 5$ and $P_n(5) = \{65533\}$). On the other hand, since
lim_{n \to \infty} A_n^*(i, j) = A_\infty^*(i, j) for any fixed \(i\) and \(j\), the function \(s(n)\) is unbounded. We have also tried all possible starting columns for all \(n \leq 10\) and there are no other patterns.

Here is some of what we know theoretically. First, \(P_n\) cannot be all of \(\mathbb{N}_n\); in other words, at least some numbers have to disappear as we move from column to column. To prove this, suppose that \(P_n = \mathbb{N}_n\). Since \(A_n(i+1, 0) = A_n(i, 1)\), the number 1 cannot appear in column \(i+1\) except in row \(n-1\), or else \(A_n(i+1, 0)\) would be repeated. Hence 1 must appear in row \(n-1\) in every column from 1 on. But the only way that \(A_n(i+2, n-1)\) gets to be 1 is for \(A_n(i+2, n-2)\) to be \(n-1\) (because 1 appears only in row \(n-1\) of column \(i+1\)). Hence \(n-1\) must appear in row \(n-2\) in every column from 2 on. Similarly, \(n-2\) must appear in row \(n-3\) in every column from 3 on. Eventually this says that 2 must appear in row 1 in every column from \(n-1\) on, which is absurd, since if 2 appears in row 1 in column \(i\), then it appears in row 0 in column \(i+1\). The “line-em-up” argument used in this proof seems useful in deriving other results as well.

Once we know at least that \(P_n \neq \mathbb{N}_n\), under what conditions can we go the whole distance and prove that \(|P_n| = 1\) (i.e., \(A_n\) stabilizes)? On the one hand, we can prove that \(|P_n| = 1\) if \(0 \notin P_n\) or \(1 \in P_n\). Our strongest result is that the standard mod-\(n\) Ackermann function stabilizes if \(n\) has a prime factor \(p\) such that \(2^{j+3} \equiv 3 \pmod{p}\) has no solutions; this is the case for \(p = 2, 3, 7, 17, 31, 41, 43\), to name the first few. From still another perspective, we can show that the two situations discussed above (and illustrated in Figures 1 and 2) are the only possible asymptotic behaviors when \(|P_n| \leq 4\) or the period is 2. Open questions abound, such as whether 1969 is the only counterexample to stability in the standard case, or how to compute \(s(n)\) efficiently.

As a final variation, we can run the Ackermann function “in reverse” to generate for each \(n\) a canonical but random-looking permutation of \(\mathbb{N}_n - \{1\}\), somewhat in the spirit of the shuffles reported on by David Gale [G]. Again we start with \(A(0, j) = j + 1\) for all \(j > 0\), but we set \(A(0, 0) = 0\). The procedure for producing column \(i+1\) from column \(i\) is
as follows: \( A(i+1,1) = A(i,0) \), and for \( j \neq 1 \), \( A(i+1,j) = A(i,k+1) \), where \( A(i,k) = j \).

The first few columns are shown in Figure 3.

\[
\begin{array}{cccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & .
\end{array}
\]

\[
\begin{array}{cccccccccccc}
8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & \ldots \\
7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 3 & \ldots \\
6 & 7 & 7 & 7 & 7 & 7 & 7 & 5 & 0 & \ldots \\
5 & 6 & 6 & 6 & 6 & 6 & 6 & 0 & 0 & 8 & \ldots \\
4 & 5 & 5 & 5 & 5 & 3 & 5 & 6 & 2 & \ldots \\
3 & 4 & 4 & 4 & 2 & 0 & 6 & 2 & 4 & \ldots \\
2 & 3 & 3 & 0 & 0 & 5 & 4 & 4 & 6 & \ldots \\
1 & 2 & 0 & 2 & 3 & 4 & 2 & 3 & 7 & \ldots \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
j = 0 & 0 & 2 & 3 & 4 & 2 & 3 & 7 & 5 & \ldots \\
i = 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots 
\end{array}
\]

**Figure 3.** The Ackermann function in reverse.

Note that each column can be obtained from the column following it by applying our original construction. It is easy to show that this function is well-defined; it gives a permutation of \( \mathbb{N} - \{1\} \) in each column that leaves \( A(i,j) = j + 1 \) for all \( j > i \). Here one might ask, for example, whether every positive integer \( j \neq 1 \) appears infinitely often in each row other than row \( j \). As of yet, we have no answers.

**References**


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