THE COMPOSITION METHOD

by

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DEDICATION

To my daughter Pamela Grossman, who was killed at the age of 6 not long before I started working on this dissertation. I would much rather have spent the time with her.
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# TABLE OF CONTENTS

Dedication ................................................................. ii
Acknowledgements ....................................................... iii
1. Introduction ............................................................. 1
2. Ehrenfeucht Games and Bounded Theories ....................... 6
3. Two Easy Examples ................................................... 14
4. Monadic Theories of Chains ......................................... 29
5. Monadic Theories of Countable Chains ........................... 37
6. A General Template ................................................... 44
7. A Final Application .................................................... 52
References ................................................................. 57
Abstract ................................................................. 59
Autobiographical Statement ............................................ 60
1. Introduction

In this dissertation, we describe a powerful decidability technique for monadic second-order theories. Monadic (second-order) logic is the extension of first-order logic that allows quantification over monadic (i.e., unary) predicates. Work in mathematical logic on monadic second-order theories found application in computer science because many properties of certain classes of structures that are of interest in computer science are expressible in this logic. Decidability in this context means that there is an effective way to tell whether or not such a property holds in any or in all of the structures in the class. Also, techniques developed to show decidability have proved to be useful in designing algorithms to tell whether or not a property holds in a given structure. For example, many NP-complete graph properties (e.g., $k$-colorability for fixed $k \geq 3$) are expressible in monadic second-order logic. Techniques used to show the decidability of the monadic second-order theory of the class of finite graphs of some fixed bounded tree-width have been used as part of efficient algorithms to test these properties on graphs in this class.

Another area of application is that of program specification for reactive systems, that is, systems whose goal is to interact continuously with their environment rather than to produce some output and terminate. Such systems are analyzed by using infinite computations as models. Propositional forms of program logics such as Temporal Logic or Computation Tree Logic are used for program specification for finite-state reactive systems. In this context, it is important to know whether or not a given formula is satisfiable, that is, if there is a model of the logic in which it is true. For many program logics, the existence of an algorithm to test satisfiability follows from known decidability results from mathematical logic for monadic second-order theories.

Two basic techniques appear in many proofs of decidability for non-trivial monadic second-order theories. The first involves automata (on finite or infinite strings or trees) and/or infinite games. It was used by Michael Rabin [Ra69] to prove
one of the most powerful and famous results in this area, the decidability of the monadic second-order theory of the infinite binary tree (also known as the monadic second-order theory of two successors, or S2S). Here we discuss the second technique, sometimes referred to as the model-theoretic technique. Because application of the model-theoretic method involves expressing structures or models as compositions of “simpler” ones, we refer to the model-theoretic technique as the *composition method*. The term “composition” actually means generalized products in the sense used by Solomon Feferman and Robert Vaught [FV59]. The Feferman-Vaught Theorem reduces the first-order theory of the composition to the first-order theories of the parts and the monadic theory of the index structure. Decidability proofs using the composition method involve reducing the monadic theory of the composition to the monadic theory of the parts and the monadic theory of the index structure.

Many of the results originally proved using automata and games have been obtained using the composition method, and this method applies in cases in which the traditional notions of automata are inappropriate. The result of Rabin mentioned above seems to be the outstanding (if not essentially the only) result in this area achievable using automata and games for which there is no known proof using the model-theoretic or composition method. The notion of composition that is part of the method is tied to that of finite automata, and to that of proposed algebraic notions of recognizability. The two decidability techniques appear to be difficult to separate clearly. An introduction to both methods as well as a discussion of related results (many of which are not mentioned here) can be found in a survey by Yuri Gurevich [Gu85].

An early version of the model-theoretic technique or composition method was used by Hans Läuchli [Lä68] to show the decidability of the weak monadic second-order theory of linear order. The method was greatly advanced by Saharon Shelah, who in [Sh75] used it to prove new decidability results along with all known decidability results about monadic theories of linear order in a uniform way. Shelah’s
work was begun with the intent of showing the decidability of the monadic theory of the real line, but the result was the development of a completely new undecidability technique that was used to show this theory to be undecidable ([Sh75] and [GS82]). Here we discuss only the decidability method. With the work by Shelah [Sh75] and the continuation with Gurevich in [Gu77], [Gu79] and [GS79], it was shown that there is a class of linearly ordered sets – the class of short modest chains – whose monadic second-order theory is decidable and is the same as that of the class of countable chains. Furthermore, the monadic theory of any non-modest chain is undecidable ([GS79] and [GS82]). So in some sense, using this technique, the decidability results for monadic theories of linear order were extended as far as possible.

Unfortunately, the techniques involved in the composition method have remained relatively unknown. Here we show that the composition method provides a uniform and illustrative approach to many decidability problems. In all of our applications, we begin with a language and a class of structures for that language. Our goal is to show that the theory of the class of structures is decidable, that is, to show the existence of an algorithm that given a sentence in the language determines whether or not that sentence is true in all the structures in the class.

In Section 2 we give the basic definitions necessary to the method; in particular, we define formal objects called bounded theories. We also discuss the relationship between these theories, which are the basic tools of the composition method, and the more familiar model-theoretic games developed by Andrzej Ehrenfeucht. In Section 3 we give two applications of the composition method, along with a general description of the technique applicable in these and other cases in which the index structure is finite. The first example consists of showing the decidability of the weak monadic theory of the infinite binary tree. Proofs of this result using automata or a generalized version of automata appear in [Do65] and [TW68]. In our second application, we show the decidability of the monadic second-order theory
of finite simple graphs of bounded tree-width. Other proofs of this result appear in [Co90] and [Se92]. This second application is novel in the sense that the composition involves some identification of individual elements in the parts. We use these applications to discuss the relationship between the composition method and algebraic notions of recognizability. In Section 4 we discuss the monadic theories of linearly ordered sets and present a basic result from [Sh75] and [Gu79], along with some examples. In Section 5 we show that the monadic second-order theory of the class of countable chains is decidable. Rabin showed the decidability of this theory via interpretation in S2S [Ra69]. This result also follows from the applications of the composition method in the work of Shelah and Gurevich ([Sh75],[Gu79], and [GS79]). Läuchli used the technique to show the decidability of the weak monadic second-order theory of the class of countable chains. (The result for arbitrary chains in [Lä68] is based on an appeal to Löwenheim-Skolem.) Here we present a proof for the full (i.e., not weak) monadic theory using the result for chains presented in Section 4. Our proof is certainly much simpler than the proof for the larger class of short modest chains, but it does give some intuition into the methods involved in the more difficult proofs of [Gu79] and [GS79].

In Section 6 we give a general template for the composition method in the form of theorems that give sufficient conditions for its application. The template is partly intended to illustrate the common elements in all of our examples, as well as the basic elements in the method itself. In Section 7 we give a final application using the template of Section 6. This last application uses the decidability of S2S [Ra69] (i.e. no new proof of the decidability of S2S is presented) to show the decidability of the monadic second-order theory of certain countably infinite graphs of tree-width less than or equal to some fixed bound. Decidability for the monadic theory of this class of graphs was shown by Bruno Courcelle in [Co89]. The example is included to show that the method can be uniformly applied for both finite and infinite graphs, and that the notion of a tree-decomposition introduced by Neil Robertson
and Paul Seymour [RS84] is a special case of the kind of decomposition of models inherent in the composition method.

There are at least a couple of directions for further research suggested by the exposition, results, and examples in this dissertation. One is to clarify the relationship between the automata-theoretic method and the composition method, partly with the aim of developing algorithms to test the satisfiability of formulas in particular program logics. Existing algorithms for this problem involve tailoring the automata-theoretic method to the specific logic and formula involved. A similar approach with the composition method alone, or along with automata, might lead to more efficient algorithms. Some use of automata may be unavoidable since for branching time logics, solving the satisfiability problem involves (or is very similar to) Rabin’s theorem for the infinite binary tree. In any case it is possible that the uniformity of the method could be exploited in applications, especially in cases in which the size or type of formula involved is in practice somehow restricted. A second direction involves re-examining the general template presented in Section 6 with the aim of providing necessary as well as sufficient conditions for its applicability. For the type of monadic second-order logic used to describe graphs in this paper (in which the universe consists of vertices and edges), it has been shown that if the theory of a class of graphs is decidable, then there is some number $k$ such that all graphs in the class have a tree-decomposition of width less than or equal to $k$ [Se91]. The success of the composition method in the case of linear orders, along with the graph applications presented here, suggests that, as in the case of graphs, it may be possible to characterize decidability for certain kinds of languages and structures in some useful way in terms of the applicability of the method, that is, in terms of the existence of some kind of decomposition.
2. Ehrenfeucht Games and Bounded Theories

Ehrenfeucht games [Eh61] have been used to give a game-theoretic interpretation for elementary equivalence of structures for first-order languages whose vocabulary consists of a finite number of relations and individual constants. A related approach involving partial isomorphisms rather than games was used by Roland Fraissé in [Fr54]. Such games and their relatives (known as pebble games) have been used to examine complexity issues in computer science. An early example of this can be found in [Im82] in which the complexity of expressing certain graph properties is studied using a slight variant of the original game. The limitation on the languages above means that there are up to logical equivalence only finitely many first-order sentences of a given quantifier depth. In this section, we define bounded theories, and discuss their relationship to Ehrenfeucht games. The notion of “finiteness” mentioned above manifests itself in the fact that the theories we define are hereditarily finite, and that associated with a given alternation type (see below) there are indeed a finite number of such theories possible. Theorems 2.2 and 2.3 appear in [Gu79]. Theorem 2.1 gives the crucial relationship between the games, bounded theories, and formulas in the language. Some of the relationships involved are stated in [Gu79] and [Gu85]. There are many similar and well-known results relating equivalent concepts [Mo76]. This one is a slightly different formulation of these in the sense that it involves the bounded theories explicitly.

Let $L$ be a first-order language. We assume that the vocabulary of $L$ consists of finitely many relation symbols and individual constants. We consider $L$ to be fixed in the remainder of this section, that is, all structures or models are structures for $L$. Please note that we use the words structure and model interchangeably. In all of our examples and in the general template of Section 6, we begin with a language and consider a particular class of structures for that language. Although they are
not mentioned, the structures in this class usually satisfy some set of axioms such as those for linear order. Our general aim is to provide an effective procedure to decide whether or not a sentence of $L$ is true in all models in this class.

In the prenex normal form of a formula of $L$, the prefix consists of alternating blocks of universal and existential quantifiers. The *alternation type* of a formula is the sequence of the lengths of the quantifier blocks. For example the alternation type of both $\forall^3\exists^5\forall^7$ and $\exists^3\forall^5\exists^7$ is $\langle 3, 5, 7 \rangle$. We use $\bullet$ to represent concatenation, so that if $\xi$ is the alternation type $\langle 3, 5, 7 \rangle$ above, $\langle 8 \rangle \bullet \xi$ is the alternation type $\langle 8, 3, 5, 7 \rangle$. We define the *blocknumber* of an alternation type $\xi$ to be the number quantifier blocks it contains.

We define the generalized version of Ehrenfeucht games as follows. Let $M$ and $N$ be two structures, and let $\phi$ be a formula in prenex normal form with alternation type $\xi = \langle \xi_1, \xi_2, \ldots, \xi_n \rangle$. The *quantifier-free type* of a tuple $\langle x_1, \ldots, x_l \rangle$ in a structure is the set of atomic formulas $\psi(v_{i_1}, \ldots, v_{i_s})$ of $L$ with variables among $v_1, \ldots, v_l$, such that $\psi(x_{i_1}, \ldots, x_{i_s})$ holds in the structure. We use the words tuple and sequence interchangeably, and for convenience of notation we sometimes write $\bar{a}$ is in $M$ meaning that $\bar{a}$ is a tuple of elements from structure $M$. Let $\bar{m} = \langle m_1, m_2, \ldots, m_l \rangle$ be a tuple in $M$, and let $\bar{n} = \langle n_1, n_2, \ldots, n_l \rangle$ be a tuple in $N$. The generalized Ehrenfeucht game associated with the alternation type $\xi$, the structures $M$ and $N$, and tuples $\bar{m}$ and $\bar{n}$, which we call $\Gamma_\xi(M, \bar{m}, N, \bar{n})$, is played between two players as follows. On the $k$th step, player I chooses one of the structures $M$ and $N$ and a tuple of $\xi_k$ elements of the chosen structure, and player II chooses a tuple of $\xi_k$ elements of the remaining structure. Let $\langle a_1, \ldots, a_s \rangle$ be the entire tuple of $\xi_1 + \xi_2 + \cdots + \xi_n$ elements chosen from $M$, and let $\langle b_1, \ldots, b_s \rangle$ be the corresponding tuple of elements in $N$. Player II wins iff the quantifier-free type of $\langle n_1, \ldots, n_l, b_1, \ldots, b_s \rangle$ is the same as that of $\langle m_1, \ldots, m_l, a_1, \ldots, a_s \rangle$. In case $\bar{m}$ and $\bar{n}$ are empty, we refer to the game merely as $\Gamma_\xi(M, N)$. In the special case when $\xi$ is empty and there are no moves in
the game, we say player II wins iff the quantifier-free type of $\bar{m}$ is the same as that of $\bar{n}$.

The usual version of the game involves alternation types of the form $1^n$ (a sequence of $n$ ones). In such games, each player at his turn picks a single element from one of the structures. A basic result in first order logic [Eh61] is that for languages of the type described above, two structures are elementarily equivalent (they satisfy the same first-order sentences) iff player II has a winning strategy in $\Gamma_{1^n}(M, N)$ for all $n$. Here we wish to investigate the relation between generalized Ehrenfeucht games and the notion of bounded theories. Given a structure $M$ and a tuple $\bar{m}$ of $l$ elements of $M$, we refer to the pair $\langle M, \bar{m} \rangle$ as an augmented structure of weight $l$. We associate with an augmented structure $\langle M, \bar{m} \rangle$ and alternation type $\xi$, a set called the $\xi$-theory of $\langle M, \bar{m} \rangle$. A bounded theory is a $\xi$-theory for any alternation type $\xi$.

We now formally define bounded theories. For empty $\xi$, we define the $\lambda$-theory (or the 0-theory) of the augmented structure $\langle M, \bar{m} \rangle$, $TH_\lambda(M, \bar{m})$ (or $TH^0(M, \bar{m})$), as the quantifier-free type of $\bar{m}$. If $\xi = k \bullet \beta$, the $\xi$-theory of $\langle M, \bar{m} \rangle$, $TH_\xi(M, \bar{m})$ equals $\{TH_\beta(M, \bar{m} \bullet \bar{a}) \mid \bar{a} \text{ is a tuple in } M \text{ and } |\bar{a}| = k\}$. When $\xi = 1^n$, we denote $TH_\xi(M, \bar{m})$ also by $TH^n(M, \bar{m})$ and refer to it as the $n$-theory of $\langle M, \bar{m} \rangle$.

There are corresponding formal $\xi$-theories that capture the idea of possible $\xi$-theories for any augmented structure $\langle M, \bar{m} \rangle$. Let $T_\lambda(l)$ (or $T^0(l)$) be the set of atomic formula with free variables among $v_1, \ldots, v_l$. Note that $TH^0(M, \bar{m})$ is a subset of $T^0(l)$. Now let $\xi = k \bullet \beta$. We define $T_\xi(l)$ as the power set of $T_\beta(l + k)$. It is easy to show by induction on the blocknumber of $\xi$ that for any augmented structure $\langle M, \bar{m} \rangle$ of weight $l$, $TH_\xi(M, \bar{m})$ is a subset of $T_\xi(l)$. We use $PT_\xi(l)$ denote the powerset of $T_\xi(l)$. The set $PT_\xi(l)$ is then the set of formally possible values for $\xi$-theories of augmented structures of weight $l$. Let $FM_\lambda(l)$ ($FM^0(l)$) be the set
of boolean combinations of formulas in $T^0(l)$. For $\xi = k \cdot \beta$, let $FM_\xi(l)$ be the set of boolean combinations of formulas of the form $\exists v_{l+1}, \ldots, \exists v_{l+k} \psi$ where $\psi$ is in $FM_\beta(l + k)$. For simplicity of notation, for a formula $\phi = \theta(v_i_1, \ldots, v_i_s)$ with free variables among $v_1, \ldots, v_l$, and a tuple $\bar{m} = \langle m_1, \ldots, m_l \rangle$ in $M$, we write $\theta(m_i_1, \ldots, m_i_s)$ merely as $\phi[\bar{m}]$.

It is important to note that for any $\xi$ and $l$, $T_\xi(l)$ is hereditarily finite and computable from $\xi$ and $l$. Since for any augmented structure $\langle M, \bar{m} \rangle$ of weight $l$, $TH_\xi(M, \bar{m})$ is a subset of $T_\xi(l)$ (i.e. is an element of $PT_\xi(l)$), $TH_\xi(M, \bar{m})$ is also hereditarily finite, and for fixed alternation type $\xi$ and weight $l$ there are a finite number of such theories possible.

**Theorem 2.1** Let $\xi$ be an alternation type, and let $\langle M, \bar{m} \rangle$ and $\langle N, \bar{n} \rangle$ be augmented structures of weight $l$ for a first order language $L$ whose vocabulary consists of a finite number of relational symbols and constants. Then the following are equivalent:

1. Player II has a winning strategy in $\Gamma_\xi(M, \bar{m}, N, \bar{n})$,

2. The $TH_\xi(M, \bar{m}) = TH_\xi(N, \bar{n})$, and

3. If $\phi$ is a formula in $FM_\xi(l)$, then $M \models \phi[\bar{m}]$ iff $N \models \phi[\bar{n}]$.

**Proof.** We first show that 1 implies 2, by showing that player I has the winning strategy whenever $TH_\xi(M, \bar{m}) \neq TH_\xi(N, \bar{n})$. It is always the case that one of the players has a winning strategy. Proof is by induction on the blocknumber of $\xi$. For empty $\xi$ and any $l$, the claim is obvious. Otherwise, let $\xi = k \cdot \beta$. We assume that 1 implies 2 for all alternation types that have a smaller blocknumber than $\xi$, in particular we assume that the desired implication holds for $\beta$. By definition, $TH_\xi(M, \bar{m}) = \{TH_\beta(M, \bar{m} \bullet \bar{a}) \mid \bar{a} \in M \text{ and } |a| = k \}$ and $TH_\xi(N, \bar{n}) = \{TH_\beta(N, \bar{n} \bullet \bar{b}) \mid \bar{b} \in N \text{ and } |b| = k \}$. Since these theories are not equal, we can without loss
of generality assume that there is an \(\bar{a} \in M\) such that \(TH_\beta(M, \bar{m} \cdot a)\) is not in \(TH_\xi(N, \bar{n})\). The winning strategy for player I consists of choosing \(M\) and \(\bar{a}\) and playing according to his winning strategy in \(\Gamma_\beta(M, \bar{m} \cdot \bar{a}, N, \bar{n} \cdot \bar{b})\) where \(\bar{b}\) is the corresponding choice of player II. The existence of a winning strategy for player I in \(\Gamma_\beta(M, \bar{m} \cdot \bar{a}, N, \bar{n} \cdot \bar{b})\) follows from the inductive hypothesis using the fact that \(TH_\beta(M, \bar{m} \cdot \bar{a})\) is not equal to \(TH_\beta(N, \bar{n} \cdot \bar{b})\).

We now show that 2 implies 1. Again proof is by induction on the blocknumber of \(\xi\), and again the case when \(\xi\) is empty is obvious. Now assume that \(\xi = k \cdot \beta\), and that \(TH_\xi(M, \bar{m}) = TH_\xi(N, \bar{n})\). Suppose the first move of player I is to choose \(\bar{a} \in M\). Then the winning strategy for II consists of choosing \(\bar{b} \in N\) so that \(TH_\beta(M, \bar{m} \cdot a) = TH_\beta(N, \bar{n} \cdot \bar{b})\) (The existence of such a choice is guaranteed by the fact that the two \(\xi\)-theories are equal.), and then following his winning strategy guaranteed by the inductive hypothesis in \(\Gamma_\beta(M, \bar{m} \cdot \bar{a}, N, \bar{n} \cdot \bar{b})\).

Finally, we show 2 \(\iff\) 3. Suppose \(TH_\xi(M, \bar{m}) = TH_\xi(N, \bar{n})\), and let \(\phi\) be a formula in \(FM_\xi(l)\). If \(\xi\) is empty, it is obvious that \(M \models \phi[\bar{m}]\) iff \(N \models \phi[\bar{n}]\), since \(\phi\) is a boolean combination of atomic formulas and the quantifier-free types of \(\bar{m}\) and \(\bar{n}\) are the same in \(M\) and \(N\) respectively. Otherwise \(\xi = k \cdot \beta\) and \(\phi\) is a boolean combination of formulas of the form \(\theta = \exists v_{l+1}, \ldots, \exists v_{l+k} \psi\) with \(\psi\) in \(FM_\beta(l + k)\). It is sufficient to show that for such \(\theta\), \(M \models \theta[\bar{m}]\) iff \(N \models \theta[\bar{n}]\). Suppose \(M \models \theta[\bar{m}]\). Then there is a \(k\)-tuple \(\bar{a}\) in \(M\) such that \(M \models \psi[\bar{m} \cdot \bar{a}]\). Since \(TH_\xi(M, \bar{m}) = TH_\xi(N, \bar{n})\), there is a \(k\)-tuple \(\bar{b}\) in \(N\) such that \(TH_\beta(M, \bar{m} \cdot \bar{a})\) is equal to \(TH_\beta(N, \bar{n} \cdot \bar{b})\). Thus, \(M \models \psi[\bar{m} \cdot \bar{a}]\) iff \(N \models \psi[\bar{n} \cdot \bar{b}]\) by the inductive hypothesis. Thus \(N \models \theta[\bar{n}]\). The case in which \(N \models \theta[\bar{n}]\) is symmetric.

We now need to show that if \(TH_\xi(M, \bar{m}) \neq TH_\xi(N, \bar{n})\), it is not the case that \(M \models \phi[\bar{m}]\) iff \(N \models \phi[\bar{n}]\) for some formula \(\phi\) in \(FM_\xi(l)\). If \(\xi\) is empty, this is clear. We proceed by induction on the blocknumber of \(\xi\), and consider the case
\[ \xi = k \cdot \beta. \] Without loss of generality, we can assume there is a \( k \)-tuple \( \bar{a} \) in \( M \) such that \( TH_\beta(M, \bar{m} \cdot \bar{a}) \) is not an element of \( TH_\xi(N, \bar{n}) \). Thus for any \( k \)-tuple \( \bar{b} \) in \( N \), if \( t = TH_\beta(N, \bar{n} \cdot \bar{b}) \), then there is a formula \( \psi_t \) in \( FM_\beta(l + k) \) such that \( M \models \psi[\bar{m} \cdot \bar{a}] \) and \( N \not\models \psi[\bar{n} \cdot \bar{b}] \). Furthermore for any other \( k \)-tuple \( \bar{c} \) of \( N \), if \( TH_\beta(N, \bar{n} \cdot \bar{c}) = t \), then \( M \models \psi[\bar{m} \cdot \bar{a}] \) and \( N \not\models \psi[\bar{n} \cdot \bar{c}] \). This follows from the fact that 2 implies 3 when \( M = N \), and \( \bar{m} \) and \( \bar{n} \) are two distinct tuples in \( N \). Let \( \theta \) be the conjunction of \( \psi_t \) for \( t \in TH_\xi(N, \bar{n}) \). (This is where it is important that there are a finite number of such \( t \).) Then the desired \( \phi \) is \( \exists v_{l+1} \cdots \exists v_{l+k} \theta \).

**Theorem 2.2** For \( \phi \in FM_\xi(l) \), the truth value of \( \phi[\bar{m}] \) where \( \bar{m} \) is an \( l \)-tuple in structure \( M \) is computable from \( TH_\xi(M, \bar{m}) \).

*Proof.* If \( \xi \) is empty, the claim is obvious since \( \phi \) is a boolean combination of atomic formulas, and \( TH^0(M, \bar{m}) \) is contains precisely those atomic formula that are “true” of \( \bar{m} \). Otherwise, in case \( \xi = k \cdot \beta \), \( \phi \) is a boolean combination of formulas of the form \( \theta = \exists v_{l+1}, \ldots, \exists v_{l+k} \psi \) with \( \psi \in FM_\beta(l + k) \). Since the truth value of each such \( \psi \) for a particular \( k \)-tuple \( \bar{a} \) of \( M \) is computable from \( TH_\beta(M, \bar{m} \cdot \bar{a}) \) and \( TH_\xi(M, \bar{m}) = \{ TH_\xi(M, \bar{m} \cdot \bar{a}) \mid \bar{a} \text{ is in } M \text{ and } |\bar{a}| = k \} \), the truth value of \( \theta[\bar{m}] \) is computable from \( TH_\xi(M, \bar{m}) \). Thus the truth value of \( \phi[\bar{m}] \), where \( \phi \) is the boolean combination of such formulas, is also computable from \( TH_\xi(M, \bar{m}) \).

In general, \( FM_\xi(l) \) includes (up to renaming of variables) all formulas in prenex normal form of alternation type \( \xi \) with \( l \) free variables. In case \( \xi = 1^n \), formulas in \( FM_\xi(0) \) include (up to renaming of variables) all sentences in prenex normal form with \( n \) quantifiers. Thus if follows from Theorem 2.1 that structures \( M \) and \( N \) are elementarily equivalent iff \( TH^n(M) = TH^n(N) \) for all \( n \geq 0 \) iff player II has a winning strategy in \( \Gamma_1^n(M, N) \) for all \( n \geq 0 \).

The following theorem states that the property of having a particular \( \xi \)-theory is expressible in \( L \).
Theorem 2.3  There is an algorithm associating a formula $\phi_t \in FM_\xi(l)$ with each $t \subseteq T_\xi(l)$ in such a way that for every augmented structure $\langle M, \bar{m} \rangle$ of weight $l$, $t = TH_\xi(M, \bar{m})$ iff $\phi_t[\bar{m}]$ holds in $M$.

Proof. If $t \subseteq T^0(l)$ then $\phi_t$ is the conjunction of (a) the formulas in $t$ and (b) the negations of the formulas in $T^0(l) \setminus t$. Suppose $t \subseteq T_\xi(l)$. Then $\phi_t$ is the conjunction of the (a) the formulas of the form $\exists v_{l+1} \cdots \exists v_{l+k}\phi_s$ where $s \in t$ and (b) the negations of the formulas of the form $\exists v_{l+1} \cdots \exists v_{l+k}\phi_s$ where $s \in T_\xi(l) \setminus t$.

The restriction placed on the language $L$ is stronger than it need be. One can relax this restriction by insisting that there be a set of pseudo-atomic formulas in variables $v_1, \ldots, v_l$ that is finite and recursive in $l$ such that there is an algorithm associating any atomic formula with a pseudo-atomic formula that is logically equivalent in all models of the theory in question. This preserves the finiteness of the $\xi$-theories, and allows application to theories with functions such that every term involving variables $v_1, \ldots, v_l$ has some kind of canonical representation and there are a finite number of such representations.

In practice the theories considered are those of a class of structures. More precisely, let $\mathcal{K}$ be a class of structures. Then $TH_\xi(\mathcal{K}) = \{TH_\xi(M) \mid M \in \mathcal{K}\}$. The theory, $TH(\mathcal{K})$, of this class of structures is the set of sentences true in all elements of $\mathcal{K}$. Thus, $TH(\mathcal{K})$ is decidable if $TH_\xi(\mathcal{K})$ is computable for any alternation type $\xi$. Obviously the definitions of $TH_\xi(\mathcal{K})$ (for all $\xi$) and $TH(\mathcal{K})$ depend on the language $L$. In some of our applications and in the general template presented in Section 6 there is one language used to describe the models to be combined, and another to describe the index structure. Our notation for theories does not indicate the language on which they are based; we assume that the dependence on the appropriate language is understood.

It may be relatively easy to compute $\{TH^0(M, \bar{m}) \mid M \in \mathcal{K} \text{ and } \bar{m} \}$ is a tuple
of \( l \) elements of \( M \) for any \( l \). In this case, one can narrow down the set of possible \( \xi \)-theories somewhat by starting with this set rather than with \( T^0(l) \). We define \( TR_\lambda(l) = TR^0(l) \) to be \( \{TH^0(M, \bar{m}) \mid M \in \mathcal{K} \text{ and } \bar{m} \text{ is a tuple of } l \text{ elements of } M\} \). Then for \( \xi = k \cdot \beta \), we define \( TR_\xi(l) \) to be the powerset of \( TR_\beta(l + k) \). If \( t \) is an element of \( TR^0(l + k) \) (or of \( PT^0(l + k) \)), we say \( s \) is the trace of length \( l \) of \( t \) if \( s = t \cap T^0(l) \). If \( t \) is a member of \( TR_\xi(l + k) \) (or of \( PT_\xi(l + k) \)) for nonempty \( \xi \), \( s \) is the trace of length \( l \) of \( t \) if \( s \) is the trace of length \( l \) of every element of \( t \). We may, if we wish, limit membership in \( TR_\xi(l) \) to those subsets of \( TR_\beta(l + k) \) all of whose members have the same trace of length \( l \). In either case, if \( \langle M, \bar{m} \rangle \) is an augmented structure of weight \( l \) with \( M \in \mathcal{K} \), then \( TH_\xi(M, \bar{m}) \in TR_\xi(l) \). It will be convenient to view elements of \( TR_\xi(l) \) or of \( PT_\xi(l) \) as ordered in some standard manner. For example, the order may be lexicographical.

The notion of bounded theories introduced in [Lä68] involved only the alternation type \( 1^n \), that is they were all \( n \)-theories for some \( n \). The extension of the concept to other alternation types and the basic result discussed in Section 4 ([Sh75] and [Gu79]) which reduces computing the \( \xi \)-theory of particular composition to computing a theory (involving the index structure) for an alternation type with the same blocknumber as \( \xi \), but much greater length, is critical to our proof extending the result in [Lä68] for the weak monadic second-order theory of countable chains to the full theory, as well as to the proof for the class of short modest chains in [Gu79] and [GS79].
3. Two Easy Examples

So far, we have not discussed either monadic theories or composition. First of all, in our applications we express the monadic second order language syntactically as a first order language in which all variables represent sets and individual variables are represented by singleton sets. We begin by considering two easy examples in which the composition involves only a finite number of structures at a time. This will not be the case in our later applications or in the general template. In this section we present the examples in what we believe it the most straightforward manner for each at first, and only later do we discuss the similarity in each case and the general idea of the composition method for simple cases such as these. In both cases we begin with a language and a class of structures or models for this language. Recall from the previous section that the truth value of any sentence in the language in prenex normal form with $n$ quantifiers in a structure $M$ can be determined from $TH^n(M)$. Thus in order to show that the theory in question is decidable it is sufficient to show that there is an algorithm that for given $n$ computes the set of $n$-theories of the structures in the class.

Many of the theorems or lemmas in this section state the existence of some algorithm or operation for each value of $n \geq 0$. Please note that in the associated proof, the construction of the desired algorithm is uniform in $n$, that is, the process of construction can be seen as taking $n$ as a parameter. Thus what is actually being shown is that there is a uniform way to construct, given $n$, the desired algorithm for that value of $n$. This is made more explicit in Theorem 3.3, but is omitted elsewhere for the sake of readability.

For our first example, we consider the weak monadic second-order theory of the infinite binary tree. The term weak indicates that quantification is allowed only over finite sets. The proof presented here is similar to, although simpler than, the
proof of the decidability of the weak monadic second-order theory of linear order in [Lä68]. Let $T$ be the infinite binary tree, which we define here as the set $\{l, r\}^*$ of all words in the alphabet $\{l, r\}$. The empty word $e$ is the root of $T$. If $X_1, \ldots, X_k$ are subsets of $T$, let $\text{root}(X_1, \ldots, X_k)$ be the $k$-tuple $(r_1, \ldots, r_k)$ such that for $1 \leq i \leq k$, $r_i = 1$ if $e$ is an element of $X_i$, and is 0 otherwise. For a subset $X$ of $T$, let $X^l = \{y \mid ly \in X\}$, and let $X^r = \{y \mid ry \in X\}$. The monadic language of two successors contains the binary predicates $\subseteq$, $\text{LC}$, and $\text{RC}$. We regard the binary tree as a model for this language: variables range over the subsets, $\subseteq$ is the usual inclusion, $\text{LC}(X, Y)$ means that there are words $x$ and $y$ such that $X = \{x\}$, $Y = \{y\}$ and $y = xl$, and $\text{RC}(X, Y)$ means that there are words $x$ and $y$ such that $X = \{x\}$, $Y = \{y\}$ and $y = xr$. The monadic theory of the binary tree is known as S2S which is an acronym for the second-order (monadic) theory of two successors. The proof for the full monadic theory [Ra69] uses automata on infinite trees and is extremely complicated, although some simplification was made in [GH82] and [Mu84] (See also [Mo84], [YY90] and [Ze94].) The result for the full theory is extremely powerful, and many other decidability results, as well as the satisfiability problem for many programming logics, reduce to it. Proofs for the weak theory have also used automata or a generalized version of automata, this time on finite trees ([Do65] and [TW68]).

In the weak monadic theory, denoted by WS2S, variables range over finite subsets. The language we use is equivalent in expressive power with respect to the model described above to to the monadic language of two successors, but it is tailored to our method of proof. Our language contains additional atomic formulas. These additional formulas are: $\text{ROOT}(X)$ which means that the root, $e$, is an element of $X$, $\text{SING}(X)$ which means that there is a word $x$ such that $X = \{x\}$, that is, $X$ is a singleton, and $\text{EMPTY}(X)$ which means that $X$ is empty.
Lemma 3.1  There is an operation $perm^n_{ij}$ for each $n \geq 0$ and $i, j \geq 1$ with $i < j$ such that for all subsets $X_1, \ldots, X_k$ of $T$ with $k \geq 2$, $1 \leq i < j \leq k$,

$$perm^n_{ij}(TH^n(T, X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k)) = TH^n(T, X_1, \ldots, X_j, \ldots, X_i, \ldots X_k).$$

Proof. Proof is by induction on $n$. For $n = 0$, $perm^0_{ij}(TH^0(T, X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k))$ can be obtained merely by interchanging variables $v_i$ and $v_j$ in each of the formulas in $TH^0(T, X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k)$. Assuming $perm^m_{ij}$ has been defined for all values of $m < n$ and appropriate values of $i, j$ and $k$, we can define the operation for $n$ and all appropriate $i, j$, and $k$ as follows.

$$perm^n_{ij}(TH^n(T, X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k)) =$$

$$perm^n_{ij}(\{TH^{n-1}(T, X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k, Y) \mid Y \subseteq T\}) =$$

$$\{perm^{n-1}_{ij}(TH^{n-1}(T, X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k, Y)) \mid Y \subseteq T\} =$$

$$\{TH^{n-1}(T, X_1, \ldots, X_j, \ldots, X_i, \ldots, X_k, Y) \mid Y \subseteq T\} =$$

$$TH^n(T, X_1, \ldots, X_j, \ldots, X_i, \ldots, X_k).$$

Lemma 3.2  There is an operation $e^n$ for each $n \geq 0$ such that for all subsets $X_1, \ldots, X_k$ of $T$,

$$e^n(TH^n(T, X_1, \ldots, X_k)) = TH^n(X_1, \ldots, X_k, \emptyset).$$

In particular $e^n(TH^n(T)) = TH^n(T, \emptyset)$.

Proof. For $n = 0$, $e^0(TH^0(X_1, \ldots, X_k)) =$

$$TH^0(X_1, \ldots, X_k) \cup \{v_{k+1} \subseteq v_i \mid 1 \leq i \leq k + 1\} \cup \{EMPTY(v_{k+1})\}.$$
Assuming that $e^m$ has been defined for $m < n$, we can let

$$e^n(TH^n(X_1, \ldots, X_k)) =$$

$$\{\text{perm}^{n-1}_{k+1,k+2}(e^{(n-1)}(TH^{n-1}(X_1, \ldots, X_k, Y))) | Y \subseteq T\} =$$

$$\{\text{perm}^{n-1}_{k+1,k+2}(TH^{n-1}(X_1, \ldots, X_k, Y, \emptyset)) | Y \subseteq T\} =$$

$$\{TH^{n-1}(X_1, \ldots, X_k, \emptyset, Y) | Y \subseteq T\} =$$

$$TH^n(X_1, \ldots, X_k, \emptyset).$$

**Theorem 3.3** There is an algorithm $\text{COMP THEORY}$ that given $n \geq 0$ computes $TH^n(T, X_1, \ldots, X_k)$ from $TH^n(T, X^l_1, \ldots, X^l_k)$ and $TH^n(T, X^r_1, \ldots, X^r_k)$ along with $(r_1, \ldots, r_k)$, where $X_1, X_2, \ldots, X_k$ are arbitrary subsets of the infinite binary tree, $T$, and $(r_1, \ldots, r_k) = \text{root}(X_1, \ldots, X_k)$.

**Proof.** Recall that $TH^n(T, X^l_1, \ldots, X^l_k)$ and $TH^n(T, X^r_1, \ldots, X^r_k)$ are the $n$-theories of the portions of the original sets in the left and right subtrees respectively. By induction on $n$, we construct algorithms $\text{COMP THEORY}^n$ such that each $\text{COMP THEORY}^n$ computes $TH^n(T, X_1, \ldots, X_k)$ from $TH^n(T, X^l_1, \ldots, X^l_k)$ and $TH^n(T, X^r_1, \ldots, X^r_k)$ along with $(r_1, \ldots, r_k)$, where $X_1, X_2, \ldots, X_k$ are arbitrary subsets of $T$, and $(r_1, \ldots, r_k) = \text{root}(X_1, \ldots, X_k)$. The construction is uniform in $n$ (i.e. we can view $n$ as a parameter of the construction) and results in the desired algorithm $\text{COMP THEORY}$. First we show the existence of $\text{COMP THEORY}^0$, that is, we show how to compute $t = TH^0(T, X_1, \ldots, X_k)$ from the 0-theories $t^l = TH^0(T, X^l_1, \ldots, X^l_k)$ and $t^r = TH^0(T, X^r_1, \ldots, X^r_k)$, along with $(r_1, \ldots, r_k)$.

For $1 \leq i, j \leq k$, $t$ contains

$\text{EMPTY}(v_i)$ if $r_i = 0$ and $\text{EMPTY}(v_i)$ is in $t^l \cap t^r$,

$\text{SING}(v_i)$ if $\text{EMPTY}(v_i)$ is in $t^l \cap t^r$ and $r_i = 1$, or if $r_i = 0$ and $\text{SING}(v_i)$ and $\text{EMPTY}(v_i)$ are in $t^l \cup t^r$,  

$t^l$ contains

$\text{EMPTY}(v_i)$ and $\text{SING}(v_i)$ if $r_i = 0$ and $\text{EMPTY}(v_i)$ is in $t^l \cap t^r$,

$\text{SING}(v_i)$ if $\text{EMPTY}(v_i)$ is in $t^l \cap t^r$ and $r_i = 1$, or if $r_i = 0$ and $\text{SING}(v_i)$ and $\text{EMPTY}(v_i)$ are in $t^l \cup t^r$,  

$t^r$ contains

$\text{EMPTY}(v_i)$ and $\text{SING}(v_i)$ if $r_i = 0$ and $\text{EMPTY}(v_i)$ is in $t^l \cap t^r$,

$\text{SING}(v_i)$ if $\text{EMPTY}(v_i)$ is in $t^l \cap t^r$ and $r_i = 1$, or if $r_i = 0$ and $\text{SING}(v_i)$ and $\text{EMPTY}(v_i)$ are in $t^l \cup t^r$. 


$v_i \subseteq v_j$ if either $r_j = 1$ and $v_i \subseteq v_j$ is in $t^l \cap t^r$, or $r_i = r_j = 0$ and $v_i \subseteq v_j$ is in $t^l \cap t^r$.

$\text{ROOT}(v_i)$ if $r_i = 1$,

$\text{LC}(v_i, v_j)$ if the conditions for $\text{SING}(v_i)$ and for $\text{SING}(v_j)$ described above are met, and also either $\text{LC}(v_i, v_j)$ is in $t^l \cup t^r$, or $r_i = 1$ and $\text{ROOT}(v_j)$ is in $t^l$, and finally

$\text{RC}(v_i, v_j)$ if the conditions for $\text{SING}(v_i)$ and for $\text{SING}(v_j)$ described above are met, and also either $\text{RC}(v_i, v_j)$ is in $t^l \cup t^r$, or $r_i = 1$ and $\text{ROOT}(v_j)$ is in $t^r$.

Now we assume that we have defined an algorithm $\text{COMPTHEORY}^m$ for $m < n$ such that for all $k \geq 0$, and subsets $X_1, \ldots, X_k$ of $T$,

$$TH^m(T, X_1, \ldots, X_k) =$$

$$\text{COMPTHEORY}^m(TH^m(T, X_1^l, \ldots, X_k^l), TH^m(T, X_1^r, \ldots, X_k^r), (r_1, \ldots, r_k)).$$

We wish to define $\text{COMPTHEORY}^n$. Let $t^l$, and $t^r$ be the $n$-theories of the left and right subtrees respectively. Note that $t^l = \{TH^{n-1}(T, X_1^l, \ldots, X_k^l, Y^l) \mid Y \subseteq T\}$ and $t^r = \{TH^{n-1}(T, X_1^r, \ldots, X_k^r, Y^r) \mid Y \subseteq T\}$. Then we have

$$TH^n(T, X_1, \ldots, X_K) = \{\text{COMPTHEORY}^{n-1}(a, b, (r_1, \ldots, r_k, 0)) \mid a \in t^l, b \in t^r\}$$

$$\cup \{\text{COMPTHEORY}^{n-1}(a, b, (r_1, \ldots, r_k, 1) \mid a \in t^l, b \in t^r\}.$$
Proof. For \( n = 0 \), \( TH^0(T) \) is empty. We now assume that for \( m < n \) it is possible to compute \( TH^m(T) \), and we wish to compute \( TH^n(T) \). First we use the operation \( e^{(n-1)} \) of Lemma 3.2 to compute \( TH^{n-1}(T, \emptyset) \) from \( TH^{n-1}(T) \), and we let \( S^0 = \{ TH^{n-1}(T, \emptyset) \} \). For \( m > 0 \), we let

\[
S^m = S^{m-1} \cup \{ \text{COMPTHEORY}^{n-1}(a, b, (1)) \mid a, b \in S^{m-1} \} \cup \\
\{ \text{COMPTHEORY}^{n-1}(a, b, (0)) \mid a, b \in S^{m-1} \}.
\]

It is easy to see that each application of \( \text{COMPTHEORY} \) in forming the sets \( S^m \) produces an element of \( TH^n(T) = \{ TH^{n-1}(T, Y) \mid Y \text{ is a finite subset of } T \} \). Since there are a finite number of elements in this set, and for all \( m \geq 0 \), \( S^m \) is contained in this set, there must be some \( m_1 \), such that for all \( m > m_1 \), \( S^m = S^{m_1} \). In particular,

\[
S^{m_1} = S^{m_1} \cup \{ \text{COMPTHEORY}^{n-1}(a, b, (1)) \mid a, b \in S^{m_1} \} \cup \\
\{ \text{COMPTHEORY}^{n-1}(a, b, (0)) \mid a, b \in S^{m_1} \}.
\]

We claim that \( S^{m_1} = TH^n(T) \). For a finite subset of \( T \), we refer to the maximum level of \( T \) on which there is an element of that subset as the depth of the subset. Assume that \( Y \) is a finite subset of minimum depth such that \( TH^{n-1}(T, Y) \) is not in \( S^{m_1} \). Then \( TH^{n-1}(T, Y^l) \) and \( TH^{n-1}(T, Y^r) \) are in \( S^{m_1} \) since the depth of either of \( Y^l \) or \( Y^r \) is less than that of \( Y \). But this is impossible, since then \( TH^{n-1}(T, Y) \) would have been added to \( S^{m_1} \) by an application of \( \text{COMPTHEORY}^{n-1} \) to these theories and either 1 or 0 depending on whether or not the root of \( T \) is in \( Y \). Therefore \( \{ TH^{n-1}(T, Y) \mid Y \text{ is a finite subset of } T \} \) is contained in \( S^{m_1} \). Since containment in the other direction also holds, the sets are equal, that is, \( S^{m_1} = TH^n(T) \).

For our second example, we consider simple (without loops or multiple edges) undirected graphs. There are two versions of monadic second-order languages for
In the first version, variables are allowed to represent vertices or sets of vertices. In the second version, the one discussed here, variables may represent either edges or vertices. The composition inherent in these structures is that provided by the notion of a tree-decomposition developed by Robertson and Seymour [RS84]. The same notion was re-discovered by Ajtai and Gurevich [AG89] and called a nostrum which is the name used here. The decidability result in this section concerns graphs of bounded tree-width, and is most naturally obtained by considering the nostrums themselves as models. The notion of the tree-width of a graph is an important one that relates to the efficiency of many graph algorithms [ALS91]. A nostrum consists of an rooted tree $N_T$ and a graph $N_G$ along with a function that assigns to each node $t$ of $N_T$ a subset (possibly empty) of the vertices of $N_G$ called the grasp of $t$ such that

1. each vertex of $N_G$ is in the grasp of some node of $N_T$,

2. every edge of $N_G$ has both of its ends in the grasp of some node, and

3. for each vertex $v$, the set of nodes of $N_T$ that grasp $v$ is connected (when $N_T$ is viewed as an undirected graph).

The width of the nostrum is the maximum cardinality of the grasp of any node in $N_T$. The tree-width of a graph $N_G$ is one less than the minimum width of any of its nostrums.

Nostrums of width $k$ are models for the language described below. We denote by $|N|$ the union of the vertex set of $N_G$, the edge set of $N_G$ and the node set of $N_T$. Our monadic second-order language for these nostrums, $L^k_N$, contains binary predicates $\subset$, $INC$, $CHILD$ and $GRASPS$, along with unary predicates $SING$, $VERT$, and $EDGE$, $NODE$ and $ROOT$. If $X$ and $Y$ are set variables representing subsets of $|N|$, $X \subseteq Y$ is ordinary set inclusion, $SING(X)$ means $X$ has exactly one
member, \emph{VERT}(X) means that all members of \(X\) are vertices, \emph{EDGE}(X) means that all members of \(X\) are edges, \emph{NODE}(X) means that all members of \(X\) are nodes, and \emph{ROOT}(X) means that \(X\) contains the root of \(N_T\). We use \emph{INC}(X,Y) to mean that \(X\) and \(Y\) are singleton sets, \(X\) contains a single edge, \(Y\) contains a single vertex, and the edge in \(X\) is incident to the vertex in \(Y\). The predicate \emph{CHILD}(X,Y) means that \(X\) and \(Y\) are sets consisting of a single node, and that the node in \(Y\) is the child of the node in \(X\). We use the predicate \emph{GRASPS}(X,Y) to denote that \(X\) is a set consisting of a single node, \(Y\) is a set consisting of a single vertex, and the vertex in \(Y\) is in the grasp of the node in \(X\). Finally our language contains the constant 0 denoting the empty set, and for a fixed \(k\), constants \(c_i\), and \(e_{ij}\) for \(1 \leq i < j \leq k\). We require that each \(c_i\) is either a singleton set containing a vertex or is empty, \(c_i \cap c_j = \emptyset\) if \(i \neq j\), and \(e_{ij}\) is the singleton set containing the edge between \(c_i\) and \(c_j\) if there is one, and is empty otherwise. Finally we require that \(\bigcup_{i=1}^{k} c_i\) equals the set of vertices in the grasp of the root. We denote the structure given by the nostrum of width \(k\), \(N\) in which \(c_i^N\) is the set represented by \(c_i\) as \((N, c_1^N, \ldots, c_k^N)\), and we refer to such a structure as a \(k\)-nostrum.

If \((A, c_{i}^{A}, \ldots, c_{k}^{A})\) and \((B, c_{i}^{B}, \ldots, c_{k}^{B})\) are two \(k\)-nostrums, \(S \subseteq \{i \mid c_{i}^{A} \neq \emptyset\text{ and }c_{i}^{B} \neq \emptyset\}\), and for all \(i\) and \(j\) in \(S\), there is an edge between \(c_{i}^{A}\) and \(c_{j}^{A}\) if and only if there is an edge between \(c_{i}^{B}\) and \(c_{j}^{B}\), we define the \(k\)-nostrum

\[
(N, c_{1}^{N}, \ldots, c_{k}^{N}) = \text{GLUE}_S((A, c_{1}^{A}, \ldots, c_{k}^{A}), (B, c_{1}^{B}, \ldots, c_{k}^{B})))
\]

as follows. Intuitively \(N_G\) is the disjoint union of the graphs \(A_G\) and \(B_G\) in which vertices in the sets \(c_{i}^{A}\) and \(c_{i}^{B}\) for \(i \in S\) have been identified. More precisely, we define an equivalence relation on the vertex sets of \(A_G\) and \(B_G\) where two distinct vertices are related if and only if for some \(i \in S\), one of them is the single vertex in \(c_{i}^{A}\), and the other is the single vertex in \(c_{i}^{B}\). We denote the equivalence class of a vertex \(v\) by \([v]\). The vertex set of \(N_G\) is the set of equivalence classes of this
Then there is an algorithm that computes \((\text{GLUE}((A, c^A_1, \ldots, c^A_k), (B, c^B_1, \ldots, c^B_k)))\) whether or not \(\text{GLUE}_S((A, c^A_1, \ldots, c^A_k), (B, c^B_1, \ldots, c^B_k))\) is defined.

**Lemma 3.5** Suppose that \((N, c^N_1, \ldots, c^N_k)\) is equal to \(\text{GLUE}_S((A, c^A_1, \ldots, c^A_k), (B, c^B_1, \ldots, c^B_k))\) and that \(X_1, \ldots, X_l\) is a arbitrary sequence of subsets of \(|N|\). Then there is an algorithm that computes \(\text{TH}^0((N, c^N_1, \ldots, c^N_k), X_1, \ldots, X_l)\) from \(\text{TH}^0((A, c^A_1, \ldots, c^A_k), X^A_1, \ldots, X^A_l)\) and \(\text{TH}^0((B, c^B_1, \ldots, c^B_k), X^B_1, \ldots, X^B_l)\).

**Proof.** We let \(T^0_G\) denote \(\text{TH}^0((N, c^N_1, \ldots, c^N_k), X_1, \ldots, X_l)\), and define \(T^0_A\) and \(T^0_B\) analogously. Recall that for each \(i\) between 1 and \(l\), there are special compatibility conditions on \(X^A_i\) and \(X^B_i\), the restrictions of the \(X_i\) to \((A, c^A_1, \ldots, c^A_k)\) and \((B, c^B_1, \ldots, c^B_k)\). We take advantage of these conditions in determining the contents of \(T^0_G\) from those of \(T^0_A\) and \(T^0_B\). It is easy to see that formulas involving only constants are in \(T^0_N\) if and only if they are in \(T^0_A\). We now consider formulas involving variables \(v_r\) and \(v_s\) for \(1 \leq r, s \leq l\) along with the constants \(c_i\) and \(e_{ij}\) for
$1 \leq i < j \leq k$. A formula of the form $v_r \subseteq v_s$, or $0 \subseteq v_r$, or $v_s \subseteq 0$ is in $T^0_N$ if and only if it is in both $T^0_A$ and $T^0_B$. Formulas of the form $c_i \subseteq v_s$ and $e_{ij} \subseteq v_s$ are in $T^0_N$ whenever they are in $T^0_A$. If $j \notin S$, a formula of the form $v_s \subseteq c_j$ is in $T^0_N$ if $v_s \subseteq 0$ is in $T^0_B$ and $v_s \subseteq c_j$ is in $T^0_A$. If $j \in S$, the formulas $v_s \subseteq c_j$ is in $T^0_G$ precisely when it is in both $T^0_A$ and $T^0_B$. Formulas of the form $v_s \subseteq e_{ij}$ are handled similarly (depending on whether or not both $i$ and $j$ are in $S$). The formulas $VERT(v_s)$, $EDGE(v_s)$, and $NODE(v_s)$ are in $T^0_N$ whenever they are in both $T^0_A$ and $T^0_B$. Formulas asserting that a variable is a singleton are more difficult to evaluate because of the “gluing” of vertices that takes place to form $N_G$. We abbreviate $(v_s \subseteq v_r) \land (v_r \subseteq v_s)$ as $v_r = v_s$. First (and most simply), if $NODE(v_s)$ is known to be in $T^0_N$, then $SING(v_s)$ is in $T^0_N$ whenever $v_s = 0$ is one of $T^0_A$ or $T^0_B$, and $SING(v_s)$ is in the other. If $VERT(v_s)$ is known to be in $T^0_G$, then $SING(v_s)$ should be in $T^0_G$ if either $v_s = 0$ is in one of $T^0_A$ or $T^0_B$, and $SING(v_s)$ is in the other, or if $j \in S$, and $v_s = c_j$ is in both $T^0_A$ and $T^0_B$. Similarly, if $EDGE(v_s)$ is known to be in $T^0_G$, then $SING(v_s)$ should be in $T^0_G$ if either $v_s = 0$ is in one of $T^0_A$ or $T^0_B$, and $SING(v_s)$ is in the other, or if $i, j \in S$, and both $v_s = e_{ij}$ and $SING(e_{ij})$ are in both $T^0_A$ and $T^0_B$.

Now we consider formulas involving $GRASPS$. Once it is determined that $SING(v_s)$, $SING(v_r)$, $NODE(v_s)$, and $VERT(v_r)$ are in $T^0_N$, then the formula $GRASPS(v_s, v_r)$ should be in $T^0_N$ whenever it is in either $T^0_A$ or $T^0_B$. If $SING(v_s)$ and $NODE(v_s)$ are in $T^0_N$, then $GRASPS(v_s, c_i)$ is in $T^0_N$ if either $i \notin S$ and $GRASPS(v_s, c_i)$ is in $T^0_N$, or if $i \in S$ and $GRASPS(v_s, c_i)$ is in either $T^0_A$ or $T^0_B$.

Finally, we look at formulas involving $INC$. Once it is determined that $SING(v_s)$, $SING(v_r)$, $EDGE(v_s)$, and $VERT(v_r)$ are in $T^0_G$, then $INC(v_s, v_r)$ should be in $T^0_G$ whenever it is in either $T^0_A$ or $T^0_B$. If $SING(v_s)$ and $EDGE(v_s)$ are in $T^0_G$, then $INC(v_s, c_i)$ is in $T^0_G$ if either $i \notin S$ and $INC(v_s, c_i)$ is in $T^0_A$, or if
\( i \in S \) and \( \text{INC}(v_s, c_i) \) is in either \( T^0_A \) or \( T^0_B \). Finally, if \( \text{SING}(v_s), \text{VERT}(v_s) \) and \( \text{SING}(e_{ij}) \) are in \( T^0_N \), then \( \text{INC}(e_{ij}, v_s) \) is in \( T^0_N \) precisely when it is in \( T^0_A \).

**Theorem 3.6** Suppose that \((N, c_N^1, \ldots, c_N^k)\) is equal to \( \text{GLUE}_S((A, c_A^1, \ldots, c_A^k), (B, c_B^1, \ldots, c_B^k)) \) and that \( X_1, \ldots, X_l \) is an arbitrary sequence of subsets of \( |N| \). Then for each \( n \geq 0 \) there is an algorithm that computes

\[
T^n_N = TH^n((N, c_N^1, \ldots, c_N^k), X_1, \ldots, X_l))
\]

from

\[
T^n_A = TH^n((A, c_A^1, \ldots, c_A^k), X_1^A, \ldots, X_l^A))
\]

and

\[
T^n_B = TH^n((B, c_B^1, \ldots, c_B^k), X_1^B, \ldots, X_l^B)).
\]

In this case, we write \( T^n_G = \text{GLUETHEORY}_S^n(T^n_A, T^n_B) \).

**Proof.** The existence of the \( \text{GLUETHEORY}_S^n \) is the subject of Lemma 3.5. We show how to construct \( \text{GLUETHEORY}_S^n \) from \( \text{GLUETHEORY}_S^{(n-1)} \). For \( n \geq 1 \), \( T^n_N = \)

\[
\{ \text{GLUETHEORY}_S^{(n-1)}(TH^{(n-1)}((A, c_A^1, \ldots, c_A^k), X_1^A, \ldots, X_l^A), Y)),
\]

\[
TH^{(n-1)}((B, c_B^1, \ldots, c_B^k), X_1^B, \ldots, X_l^B), Z) )) | Y \subseteq |A|, Z \subseteq |B|,
\]

and \( Y \) and \( Z \) are compatible with respect to \( S \} \).

Since it is possible to determine the compatibility of the appropriate sets for pairs of elements of \( T^n_A \) and \( T^n_B \), this set is computable from \( T^n_A \) and \( T^n_B \).
We wish to find a class of basic structures from which we can construct all $k$-nostrums using our gluing operations. Let $B_k$ be the class of $k$-nostrums whose tree part consists of a single node. Then for each $n \geq 0$, it is possible to compute $TH^n(B_k)$. Let $T^n$ be the smallest set of $n$-theories that contains $TH^n(B_k)$, and for each subset $S$ of $\{1, \ldots, k\}$, contains $GLUETHEORY^n_S(T^n_A, T^n_B)$ whenever $T^n_A$ and $T^n_B$ are in $T^n$, and are theories of nostrums $(A, c^A_1, \ldots, c^A_k)$ and $(B, c^B_1, \ldots, c^B_k)$ for which $GLUES((A, c^A_1, \ldots, c^A_k), (B, c^B_1, \ldots, c^B_k))$ is defined.

**Theorem 3.7** $T^n = \{TH^n(N, c^N_1, \ldots, c^N_k) \mid N \text{ is a } k \text{-nostrum}\}$.

**Proof.** The fact that $T^n$ is contained in the second set follows from the fact that the glue operation on $k$-nostrums produces a $k$-nostrum. To see that containment in the other direction holds, first note that $T^n$ certainly contains the $n$-theory of any $k$-nostrum whose tree part consists of a single node. Now suppose $(N, c^N_1, \ldots, c^N_k)$ is a $k$-nostrum whose tree part contains more than one node. Choose a node $t$ adjacent to the root, and consider the two connected components of the tree that result from removing the edge joining the root and $t$. Let $S = \{i \mid c^G_i \text{ represents a vertex in the grasp of both the root and } t\}$. Let $A$ be the $k$-nostrum formed by the connected component containing the root of $N_t$. The graph $A_G$ is the subgraph of $N_G$ induced by the vertices in the grasps of nodes in this component, and $c^A_i = c^N_i$ for $1 \leq i \leq k$. Let $B$ a $k$-nostrum formed by the other component such $c^B_i$ represents the same vertex as $c^A_i$ for $i \in S$. It is easy to see that $TH^n(((N, c^N_1, \ldots, c^N_k))) = GLUETHEORY^n_S(TH^n(((A, c^A_1, \ldots, c^A_k))), TH^n(((B, c^B_1, \ldots, c^B_k))))$. Thus an argument based on induction on the number of nodes can be used to show containment in the other direction.

Note that since there are a finite number of possible $n$-theories of $k$-nostrums, and since the constructions of the algorithms $GLUETHEORY^n_S$ we have described are uniform in $S$ and $n$, it is possible, given $n$, to compute the set $T^n$. Thus we have shown the following.
Theorem 3.8  The $L_k^N$ theory of $k$-nostrums is decidable, that is, there is an algorithm that given $n \geq 0$, computes \{TH^n(N,c^N_1,\ldots,c^N_k) \mid N \text{ is a } k\text{-nostrum}\}

We denote by $L_k^G$ the monadic second-order language for graphs that is the subset of $L_k^N$ consisting of the predicates $EDGE$, $VERT$, $SING$, $\subseteq$, and $INC$, and the constant 0. Then we have the following corollary to Theorem 3.8.

Corollary 3.9  For each $k$, the $L_k^G$ theory of the class of finite simple undirected graphs of tree width less than $k$ is decidable.

Note that in the second example, we have introduced an operation on structures in the language and shown the existence of a corresponding operation on the theories of these structures. Although this is not explicitly done in the first example, one can present this example in the same manner in which case the operation on structures would involve attaching two structures as the left and right subtrees of a new root. Furthermore, although it has not been made explicit in the our first examples, the operations actually involve augmented structures of the same weight. For example, our $GLUES$ operation can easily be extended to one on augmented structures in which the $i$th set is (informally) the union of the $i$th sets in the operands with appropriate vertices and edges identified. This operation is defined only if the $i$th sets in each are the operands are compatible with respect to $S$. Since these were the first examples, we described the operations themselves as simply as possible. Note, however, that in the proofs Theorems 3.3 and 3.6, the operation on augmented structures is implicit.

In both of the examples of this section, the operations on (augmented) structures involve only a finite number of structures at a time. This corresponds to our notion of composition when the index is finite. For the examples in this section and many others like them, a simple description of the composition method can be
given. This description illustrates the relationship between bounded theories and other notions of recognizability.

Suppose we wish to compute $\text{TH}_\xi(\mathcal{K})$ where $\mathcal{K}$ is a class of structures or augmented structures. Suppose also that there is a set of basic structures, and a set of operations of structures such that each element of $\mathcal{K}$ is the value of a term involving the basic structures and these operations. We do not require that these operations be defined on all tuples of structures of the appropriate length. (In the second example the basic structures are the elements of $\mathcal{B}_k$.) Suppose that to each operation $\text{OPST}$ on structures there corresponds an operation $\text{OPTH}_\xi$ on $\xi$-theories such that $\text{OPTH}_\xi(t_1, \ldots, t_k)$ is defined iff there are structures $M_1, \ldots, M_k$ in $\mathcal{K}$ with $\text{TH}_\xi(M_i) = t_i$ such that $\text{OPST}(M_1, \ldots, M_k)$ is defined, and for any such $M_1, \ldots, M_k$, $\text{OPTH}_\xi(t_1, \ldots, t_k) = \text{TH}_\xi(\text{OPST}(M_1, \ldots, M_k))$.

Then we have two algebras, one of (augmented) structures and one of $\xi$-theories and a natural homomorphism between the two in which the image of a structure is its $\xi$-theory. Furthermore the second algebra is finite. In order to extend the notion of a recognizable subset (i.e. recognized by some finite automaton) to structures such as graphs for which there is no traditional notion of finite automaton, an algebraic notion of recognizability has been introduced. According to this algebraic characterization of recognizability, a class $\mathcal{L} \subseteq \mathcal{K}$ would be recognizable (with respect to the algebra involving the operations $\text{OPST}$) if it is the inverse image of a finite set under a homomorphism to a finite algebra. Note that if the conditions above are met, the inverse image of any set of $\xi$-theories is recognizable. Also, note that any subclass of $\mathcal{K}$ defined by a formula whose truth value is determined by these $\xi$-theories is then recognizable since it is the inverse image under this homomorphism of the theories in which the formula holds. For finite strings the algebraic notion of recognizability with respect to the operation of concatenation and the notion of recognizability by finite automata are equivalent. Also in this
case the notions of recognizability and definability in monadic second order logic coincide in the sense that a language is recognizable if and only if it is definable in monadic second order logic ([Bü60], [El61]). Since there are corresponding operations of ξ-theories for each ξ, this means that in this situation, the only relevant finite algebras are the algebras consisting of ξ-theories.

But where does decidability come in? Suppose also that we can compute the ξ-theories of the our basic structures, that we can tell for each tuple of ξ-theories and operation on these theories whether or not this operation is defined on that tuple, and that when it is defined we can compute it. Then we can compute $TH_\xi(\mathcal{K})$ by computing the smallest set of ξ-theories that contains the ξ-theories of the basic elements and if closed under our operation on theories. If there is an algorithm that can do this for any given alternation type, then the theory is decidable.
4. Monadic Theories of Chains

In this section we begin our discussion of the monadic theories of linear order. Various versions of the results presented in this section can be found in [Sh75], [Gu79], and [Gu85]. The full proof for the class of short modest chains is in [Gu79] and [GS79]. Some of the definitions in this section are not applied until the treatment of the class of countable linearly ordered sets in the next section, but they are given here for the sake of completeness. A chain is a linearly ordered set. A non-empty subset of a chain is convex if whenever it contains two points of the chain, it also contains all points of the chain between them. The open sets in each chain are the unions of intervals of the form \((a, b)\). For convenience, we consider any open convex subset to be an interval. Two points in a chain form a jump if they are different and there is no point between them. A chain is densely ordered if it has no jumps. A subset \(X\) of a chain \(M\) is dense in \(M\) if between every two points of \(M\), there is a point of \(X\). An equivalence relation \(E\) on a chain \(M\) is a congruence if every equivalence class is convex. The equivalence classes are naturally ordered, and the quotient chain is denoted by \(M/E\). One of the chains discussed in this section is the ordinal \(\omega\). We use \(\omega^*\) to denote the dual of \(\omega\), that is, \(\omega\) with the ordering reversed.

Let BOOL be the first-order language of boolean algebras containing all the usual boolean operations and the equality predicate. Our language for chains, \(L_c\), is the monadic language of order that is obtained from BOOL by adding the predicate \(\leq\), and constants \(\text{min}^*\) and \(\text{max}^*\). Every chain gives a standard model for \(L_c\) in the following way. Variables range over subsets, and \(X \leq Y\) iff \(X = \{x\}\) and \(Y = \{y\}\) and \(x \leq y\) in \(M\). The constant \(\text{min}^*\) is the set whose only element is the minimum element of \(M\) if there is one, and is the empty set otherwise. Similarly, \(\text{max}^*\) is the singleton set that contains the maximum element of \(M\) if there is one, and is
empty otherwise. Any term involving only variables \( v_1, \ldots, v_l \) can be written in the canonical form \( \bigcup_j \bigcap_i s_{ij} \) where \( s_{ij} \) is either \( v_k \) or its complement for some value of \( k \) between 1 and \( l \), or is a boolean constant. Thus, we can consider bounded theories as defined in section 1. Formulas in these theories will be of the form \( \sigma = \tau \) or \( \sigma \leq \tau \) in which each of \( \sigma \) and \( \tau \) are terms in canonical form.

We call augmented structures for \( L_c \) augmented chains or acs. A subac of \( M \) is an ac \( \widetilde{X} = \langle X, P|X \rangle \) where \( X \) is a convex subset of \( M \) and \( P|X \) is the sequence of intersections of each set in \( P \) with \( X \). The subac \( \widetilde{X} \) is an interval if \( X \) is an interval of \( M \). Two acs \( \langle M, P_1, \ldots, P_l \rangle \) and \( \langle N, S_1, \ldots, S_l \rangle \) are isomorphic (written \( \langle M, P_1, \ldots, P_l \rangle \cong \langle N, S_1, \ldots, S_l \rangle \)) if there is an order preserving isomorphism \( h \) mapping from \( M \) to \( N \) such that \( h(P_i) = S_i \) for \( 1 \leq i \leq l \). Note that isomorphic acs have the same \( \xi \)-theory for every alternation type \( \xi \).

**Lemma 4.1** The set \( \{ TH^0(\widetilde{M}) : \widetilde{M} \text{ is an ac of weight } l \} \) is computable from \( l \).

**Proof (Sketch).** The 0-theory of an ac of weight \( l \) is determined by whether or not the underlying chain of the ac has a maximum or a minimum, and by various parameters associated with the values of terms of the form \( \bigcap_{i=1}^l s_i \) where for \( 1 \leq i \leq l \), \( s_i \) is either \( v_i \) or its complement. These terms may be either empty or nonempty, and each of the nonempty terms may or may not be a singleton, and may or may not contain the maximum or minimum of the chain. The terms which are singletons are in some relative order, and it may or may not be the case that the union of all of these terms equals the entire chain. For each consistent choice of the parameters described above, there is an ac of weight \( l \) with the theory determined by these choices, and, conversely, any 0-theory determines all of these choices.

An ac \( \widetilde{M} = \langle M, P \rangle \) is the sum of acs \( \widetilde{M}_i \) with respect to a chain \( I \) if there is a congruence \( E \) on \( M \) and an order preserving isomorphism \( f \) from \( I \) to \( M/E \) such that for each \( i \in I \), \( \widetilde{M}_i \cong \langle f(i), P|f(i) \rangle \). If \( I = \{1, 2\} \) with the natural order (i.e \( I \)}
is a two point chain) we write \( \widetilde{M} = \widetilde{M}_1 + \widetilde{M}_2 \). If every \( \widetilde{M}_i \) is isomorphic to a single ac \( \widetilde{N} \), we say that \( \widetilde{M} \) is the product of \( \widetilde{N} \) and \( I \) and write \( \widetilde{M} = I \cdot \widetilde{N} \). If \( \langle M, P \rangle \) is the sum of acs \( \widetilde{M}_i \) with respect to the chain of rationals \( \mathcal{Q} \), and there is a finite set of acs \( \{ \widetilde{N}_1, \ldots, \widetilde{N}_k \} \) such that for \( 1 \leq j \leq k \), the set \( E_j = \{ i \mid \widetilde{M}_i = \widetilde{N}_j \} \) is dense in \( \mathcal{Q} \) and \( \mathcal{Q} = \bigcup_{j=1}^{k} E_j \), we say \( \widetilde{M} \) is the shuffle of the acs \( \{ \widetilde{N}_1, \ldots, \widetilde{N}_k \} \). This is the same operation described for chains by Läuchli in [Lä68].

Intuitively, we wish to compute the theory of a sum \( \widetilde{M} = \langle M, P \rangle \) with respect to a congruence \( E \) from knowledge of theories of the equivalence classes themselves, and of the sets (of the index structure) that index summands with a given theory. With this in mind, we define the sequence \( E(\xi, P) = \langle E_t(\xi, P) \mid t \in TR_\xi(l) \rangle \) where \( l \) is the length of \( P \) and \( E_t(\xi, P) = \{ X \in M/E : TH_\xi(X, P | X) = t \} \).

**Lemma 4.2** Let \( \widetilde{M} = \langle M, P \rangle \) be a sum of acs with respect to congruence \( E \). Then \( TH^0(M, P) \) is computable from \( TH^0(M/E, E(\lambda, P)) \).

**Proof.** First note that \( M \) has a maximum (minimum) if and only if \( M/E \) has a maximum (minimum) and the equivalence class that is the maximum (minimum) in \( M/E \) has a maximum (minimum). For example, \( M \) has a maximum if \( TH^0(M/E, E(\lambda, P)) \) does not contain \( \text{max}^* \cap v_t = 0 \) for some \( t \), and \( \text{max}^* = 0 \) is not an element of \( t \). In this case, the maximum element of \( M \) is in an equivalence class in \( E_t(\lambda, P) \) (which for convenience we write as \( E_t \)). Note that \( E_t \) is a singleton if and only if \( v_t \leq v_t \) is in \( TH^0(M/E, E(\lambda, P)) \). The basic idea is to compute the truth of a formula from information about the intersection of each of the terms involved in the formula with each equivalence class. In the case in which \( M \) has a maximum or minimum, extra attention must be given to terms containing
min* or max* since the evaluation of the intersection of these terms with a given equivalence class depends on whether or not the equivalence class contains either the maximum or minimum or both.

In order to compute $TH^0(M, P)$, it is sufficient to consider formulas of the form $\alpha = \tau$ and $\alpha \leq \tau$ where $\alpha$ and $\tau$ are terms in variables $v_1, \ldots, v_l$ where $l$ is the length of $P$. We first consider formulas of the form $\alpha = \tau$. Let $\alpha_0$ be the result of substituting 0 for each occurrence of $\max^*$ or $\min^*$ in $\alpha$, and let $\alpha_1$ ($\alpha_2$) be the result of substituting 0 for each occurrence of $\min^*$ ($\max^*$) in $\alpha_t$. If $M$ has no maximum or minimum, then $\alpha = \tau$ is in $TH^0(M, P)$ if and only if $\alpha_0 = \tau_0$ is in every $t$ such that $E_t$ is not empty (i.e. $v_t = 0$ is not in $TH^0(M/E, E(\lambda, P))$). If $M$ has a either a maximum or a minimum or both, and it is not the case that the maximum and minimum are in an equivalence class in the same $E_t$, then $\alpha = \tau$ is in $TH^0(M, P)$ if and only if $\alpha_0 = \tau_0$ is in any $t$ such that $E_t$ is not empty and neither the maximum or minimum is in a class in $E_t$, $\alpha_1 = \tau_1$ is in the theory $t_1$ such that the maximum is in a class in $E_{t_1}$, $\alpha_2 = \tau_2$ is in the theory $t_2$ such that the minimum is in a class in $E_{t_2}$, $\alpha_0 = \tau_0$ is in $t_1$ if $E_{t_1}$ is not a singleton, and $\alpha_0 = \tau_0$ is in $t_2$ if $E_{t_2}$ is not a singleton. If $M$ has both a maximum and a minimum element and both are in a class in $E_{t_1}$ where $E_{t_1}$ is a singleton, then $\alpha = \tau$ is in $TH^0(M, P)$ if and only if $\alpha = \tau$ is in $t_1$ (here there is only one equivalence class). Finally, if $M$ has both a maximum and a minimum element and both are in a class in $E_{t_1}$ where $E_{t_1}$ is not a singleton, then $\alpha = \tau$ is in $TH^0(M, P)$ if and only if $\alpha_0 = \tau_0$ is in $t$ for all $t \neq t_1$ such that $E_t$ is not empty, $\alpha_1 = \tau_1$ and $\alpha_2 = \tau_2$ are in $t_1$, and $\alpha_0 = \tau_0$ is in $t_1$ if $\min^* \cup \max^* = v_{t_1}$ is not in $TH^0(M/E, E(\lambda, P))$.

Note that extra considerations were made above for cases such as the one in which the maximum element of $M$ is in a class in $E_t$, and $E_t$ is not a singleton. Similar considerations must be made to determine whether or not a term $\alpha$ represents a singleton in $M$. The intersection of the subset of $M$ represented by term $\alpha$ with
one class in $E_t$ may be equal to the term $\alpha_0$ in $t$, while the intersection with another class in $E_t$ is equal to $\alpha_1$. Here we consider only the case in which $M$ has neither a maximum nor a minimum element. The remaining cases are straightforward. If $M$ has neither a maximum nor a minimum, then $\alpha \leq \tau$ is in $TH^0(M, P)$ if $\alpha_0 = 0$ is in each $t$ with $E_t \neq 0$ except one, say $t_1$, $\tau_0 = 0$ is in each $t$ with $E_t \neq 0$ except one, say $t_2$, $\alpha_0 \leq \alpha_0$ is in $t_1$ and $\tau_0 \leq \tau_0$ is in $t_2$, both $v_{t_1}$ and $v_{t_2}$ are singletons, and either $t_1 = t_2$ and $\alpha_0 \leq \tau_0$ is in $t_1$, or $t_1$ and $t_2$ are not equal and $v_{t_1} \leq v_{t_2}$ is in $TH^0(M/E, E(\lambda, P))$.

Let $\xi$ be an alternation type. We define the alternation type $T(\xi, l)$ to be $\lambda$ if $\xi = \lambda$ and to be $|TR_\beta(l + k)| \cdot T(\beta, l + k)$ if $\xi = k \cdot \beta$. Note that the blocknumber of sequence $\xi$ and that of $T(\xi, l)$ are the same for any $l$. (In Section 6, we use an analogous definition of $T(\xi, l)$ in terms of $PT_\beta(l + k)$ rather than $TR_\beta(l + k)$.)

**Theorem 4.3** Let $\widehat{M} = \langle M, P \rangle$ be a sum of acs with respect to congruence $E$. Then for any alternation type $\xi$, it is possible to compute $TH_\xi(M, P)$ from $TH_\eta(M/E, E(\xi, P))$ where $\eta = T(\xi, l)$ and $l$ is the length of $P$. In this case, we say $TH_\xi(M, P)$ is the prototype of $TH_\eta(M/E, E(\xi, P))$.

**Proof.** The case of empty $\xi$ is handled by Lemma 4.2. Now assume that $\xi = k \cdot \beta$. We wish to compute $TH_\xi(M, P)$ which is equal to the set

$$S = \{TH_\beta(M, P \cdot Q) \mid |Q| = k\}.$$  

By the inductive hypothesis $S$ is computable from

$$S_1 = \{TH_{T(\beta, l+k)}(M/E, E(\beta, P \cdot Q)) \mid |Q| = k\}.$$  

Recall that $\eta = |TR_\beta(l + k)| \cdot T(\beta, l + k)$. The set $S_1$ is computable from

$$S_2 = \{TH_{T(\beta, l+k)}(M/E, E(\xi, P) \cdot E(\beta, P \cdot Q)) \mid |Q| = k\}.$$  

But $S_2$ is computable from

$$TH_\eta(M/E, E(\xi, P)) = \{TH_{T(\beta, l+k)}(M/E, E(\xi, P) \bullet R) \mid |R| = |TR_\beta(l+k)|\}$$

since $S_2$ consists of exactly those elements in this set for which the nonempty sets of $R$ form a partition of $M/E$, and for which $t$ is an element of $s$ whenever $R_t \cap E_s(\xi, P)$ is not empty.

**Corollary 4.4** Suppose $\tilde{M} = \tilde{M}_1 + \tilde{M}_2$. There is an algorithm that for any given alternation type $\xi$, computes $t = TH_\xi(\tilde{M})$ from $t_1$ and $t_2$ where for $i = 1, 2$, $t_i = TH_\xi(\tilde{M}_i)$. If $t$ is the theory that results from applying this algorithm to $t_1$ and $t_2$, we write $t = t_1 \oplus t_2$.

*Proof.* This follows from the decidability of the monadic second-order theory of two-point chains. Please note that it is also possible to construct a proof along the lines of those found in the previous section.

**Corollary 4.5** Suppose that $\tilde{M}$ is the sum of acs $\tilde{M}_i$ with respect to $\omega$ ($\omega^*$) such that for $i \in \omega$ ($\omega^*$), $TH_\xi(\tilde{M}_i) = t_1$. Then it is possible to compute $t = TH_\xi(\tilde{M})$ from $t_1$ and $TH_\eta(\omega)$ ($TH_\eta(\omega^*)$) where $\eta = T(\xi, l)$. In this case, we write $t = MULT(TH_\eta(\omega), t_1)$ (or $t = MULT(TH_\eta(\omega^*), t_1)$).

*Proof.* By Theorem 3.2, $t$ is computable from $TH_\eta(M/E, E(\xi, P))$ where $M/E$ is isomorphic to $\omega$ ($\omega^*$), $E_{t_1}(\xi, P) = M/E$, and $E_t(\xi, P)$ is empty for $t \neq t_1$. This theory is computable from $TH_\eta(M/E)$ which is the same as $TH_\eta(\omega)$ ($TH_\eta(\omega^*)$).

**Theorem 4.6** The monadic second order theory of finite chains is decidable.

*Proof.* For any $n$, it is certainly possible to compute the $n$-theory of the singleton chain. Again, since there are only a finite number of possible $n$-theories, it is possible to compute the smallest set of theories that contains this theory and is closed under the operation $\oplus$. It is easy to see that this is the desired set of $n$-theories.
**Theorem 4.7** The monadic second order theory of $\omega$ is decidable.

**Proof.** Suppose that $X$ is a sequence of length $l$ of subsets of $\omega$, and $\xi$ is any alternation type. Let $C_{(\xi,l)} = \{ TH_{\xi}(M,P) \mid M$ is a finite chain and $P$ is a sequence of $l$ subsets of $M \}$. We identify each element of $C_{(\xi,l)}$ with a color, and for $x$ and $y$ in $\omega$, we color the set $\{ x,y \}$ with the color corresponding to $TH_{\xi}([x,y),X]\{x,y\})$.

By Ramsey’s theorem [Ra29], there is an infinite homogeneous subset $J$ of $\omega$, that is, for all $x < y$ in $J$, $\{x,y\}$ has the same color. Let $b = TH_{\xi}([x,y),X]\{x,y\})$ for such a pair $x$ and $y$ in $J$, and let $a = TH_{\xi}([0,x_1),X]\{0,x_1\})$ for some $x_1 \in J$. Then $TH_{\xi}(\omega,X) = a \bigoplus (MULT(TH_{(\xi,l)}(\omega),b))$.

Thus for any alternation type $\xi$ and length $l$, $\{ TH_{\xi}(\omega,X) \mid X$ is a sequence of length $l$ of subsets of $\omega \} = \{ a \bigoplus (MULT(TH_{(\xi,l)}(\omega),b)) \mid a$ and $b$ are in $C_{(\xi,l)} \}$. For any $\xi$ and $l$, we can compute the set $C_{(\xi,l)}$ by Theorem 4.6. For $b \in C_{(\xi,l)}$, we could compute $a \bigoplus MULT(TH_{(\xi,l)}(\omega),b)$ if we could compute $TH_{T(\xi,l)}(\omega)$. This seems the kind of thing we are trying to compute in the first place, but not quite. First of all, if $\xi$ is the empty string there is no problem, that is, we can surely compute $TH^0(\omega)$. This means that for any $l$, we can compute $\{ TH^0(\omega,X) \mid X$ is a sequence of length $l$ of subsets of $\omega \}$.

Now we assume that for all lengths $k$ and alternation types $\eta$ of blocknumber less than that of $\xi$, we can compute $\{ TH_{\eta}(\omega,X) \mid X$ is a sequence of length $k$ of subsets of $\omega \}$. Suppose that we wish to compute $\{ TH_{\xi}(\omega,X) \mid X$ is a sequence of length $l$ of subsets of $\omega \}$ where $\xi = k \cdot \beta$. As noted above, we can do this if we can compute $TH_{T(\xi,l)}(\omega) = \{ TH_{T(\beta,l+k)}(\omega,S) \mid S$ is a sequence of subsets of $\omega$ of length $|TR_{\beta}(l+k)| \}$. Now apply the inductive hypothesis with $\eta = T(\beta,l+k)$, and $k = |TR_{\beta}(l+k)|$.

**Corollary 4.8** Suppose that $\widetilde{M}$ is the sum of acs $\widetilde{M}_i$ with respect to $\omega$ ($\omega^*$) and for all $i \in \omega$ ($\omega^*$), $TH_{\xi}(\widetilde{M}_i) = t_1$. There is an algorithm that (for arbitrary $\widetilde{M}$, $\xi$
meeting the conditions above) computes $t = TH_\xi(\tilde{M})$ from $t_1$. If $t$ is the theory computed by this algorithm applied to $t_1$, we write $t = \bigotimes_\omega t_1 (t = \bigotimes_{\omega^*} t_1).

Proof. The proof above for $\omega$ is easily adapted to show the decidability of the monadic second-order theory of $\omega^*$. This means that $TH_\eta(\omega)$ and $TH_\eta(\omega^*)$ are computable for any alternation type $\eta$. Now use Corollary 4.5.
5. Monadic Theories of Countable Chains

In section 4 we computed the set of $n$-theories of finite chains by computing the smallest set of theories containing the $n$-theory of the singleton chain and closed under the addition operation defined on theories. We wish to show that a similar technique can be used to compute the set of $\xi$-theories of the class of countable chains; that is, we wish to calculate $\text{TH}_\xi(\mathcal{K}) = \{ \text{TH}_\xi(M) \mid M \in \mathcal{K} \}$ where $\mathcal{K}$ is the class of countable linear orders. We call an ac $\widehat{M} = \langle M, P \rangle$ countable if $M$, its underlying chain, is countable.

We have previously defined three operations ($\bigoplus$, $\otimes_\omega$, and $\otimes_{\omega^*}$) on theories. Each of these operations corresponds to an operation on acs. One operation on acs, the shuffle, has not yet been given a corresponding operation on theories. We wish to introduce this operation now as it is the one remaining operation necessary to compute the set of $\xi$-theories of countable chains. In the previous sections we have introduced notation for operations along with a description of how to compute them. Here we proceed differently. We first define the shuffle operation on theories and later in this section we show that it is possible to compute it.

A set of pairwise disjoint subsets of $\mathcal{Q}$, $\{P_1, \ldots, P_k\}$ form a dense partition of $\mathcal{Q}$ if each $P_i$ is dense in $\mathcal{Q}$, and $\mathcal{Q} = \bigcup_{i=1}^{k} P_i$. If the condition that each $P_i$ is dense in $\mathcal{Q}$ is replaced by the condition that each $P_i$ is either empty or dense in $\mathcal{Q}$, we say that the sets $P_1, \ldots, P_k$ form a partition of $\mathcal{Q}$ to dense or empty subsets. If $\langle M, P_1, \ldots, P_k \rangle$ and $\langle N, S_1, \ldots, S_k \rangle$ are countable acs, $M$ and $N$ are densely ordered and without endpoints (i.e. they are isomorphic to $\mathcal{Q}$), the sets $P_1, \ldots, P_k$ form a partition of $M$ to dense or empty subsets, the sets $S_1, \ldots, S_k$ form a partition of $N$ to dense or empty subsets, and $P_i = \emptyset \iff S_i = \emptyset$ for $1 \leq i \leq k$, then $\langle M, P_1, \ldots, P_k \rangle \cong \langle N, S_1, \ldots, S_k \rangle$. In particular for all alternation types $\xi$, $\text{TH}_\xi(M, P_1, \ldots, P_k) = \text{TH}_\xi(N, S_1, \ldots, S_k)$. Furthermore, if $I$ is an interval
of $M$, the sets $P_i|I$ form a partition of $I$ to dense or empty subsets, and for all $\xi$, $TH_{\xi}(I, P|I) = TH_{\xi}(M, P)$. Thus we call a countable ac $\langle M, P \rangle = \langle M, P_1, \ldots, P_k \rangle$ such that $M$ is densely ordered and without endpoints, and the sets $P_1, 1 \leq i \leq k$, form a partition of $M$ to dense or empty subsets a uniform ac. Note that any uniform ac is isomorphic to a uniform ac of the form $\langle Q, R \rangle$ for some sequence $R$ of subsets of the rationals.

Let $\{t_1, \ldots, t_k\}$ be a set of $\xi$-theories of acs of length $l$. We define the shuffle of $\{t_1, \ldots, t_k\}$ to be the prototype of $TH_T(\xi, l)(Q, R)$ where $R = \langle R_t \mid t \in TR_{\xi}(l) \rangle$, $\langle Q, R \rangle$ is uniform, and $R_t$ is empty if and only if $t \notin \{t_1, \ldots, t_k\}$. The comments on uniform acs above show that the shuffle of theories is well defined in the sense that any $k$ sets used to give the dense partition of $Q$ would give rise to the same theory. If $t \in TR_{\xi}(l)$ is the shuffle of the theories in $\{t_1, \ldots, t_k\}$, then we write $t = \sigma(\{t_1, \ldots, t_k\})$. Then the following lemma easily follows from the definition of the shuffle of theories and Theorem 4.3.

**Lemma 5.1** If $\widetilde{M}$ is the shuffle of the set of acs $\{\widetilde{N}_1, \ldots, \widetilde{N}_s\}$, and $\{t_1, \ldots, t_k\} = \{TH_{\xi}(\widetilde{N}_i) \mid 1 \leq i \leq s\}$, then $TH_{\xi}(\widetilde{M}) = \sigma(\{t_1, \ldots, t_k\})$.

**Theorem 5.2** Let $\mathcal{K}'$ be a class of countable acs of weight $l$ closed under subacs. Let $\mathcal{S}$ be a set of $\xi$-theories that contains the $\xi$-theory of every one point ac in $\mathcal{K}'$, and is closed under the operations $\oplus$, $\otimes$, $\otimes_{\omega^*}$, and $\sigma$. Then $TH_{\xi}(\mathcal{K}') \subseteq \mathcal{S}$.

**Proof.** Let $\langle M, P \rangle$ be an ac in $\mathcal{K}'$. We say a subac of $\langle M, P \rangle$ is good if its $\xi$-theory is in $\mathcal{S}$. We wish to show that $\langle M, P \rangle$ is good. This is certainly true if $M$ is a singleton chain. Otherwise define an equivalence relation $E$ on $M$ such that two points $a$ and $b$ in $M$ with $a \leq b$ are related if every convex set contained in $[a, b]$ gives rise to a subac that is good.

First we show that a subac $\langle X, P | X \rangle$ of $\langle M, P \rangle$ is good if every two points of $X$
are related by $E$. Without loss of generality we can assume that $X$ has a first point $a$, but no last point. (The case in which $X$ has a last and no first point is symmetric. Also if $X$ has neither a first nor a last point, it is equal to the sum of chains, $B+G+D$ where $D$ has a first and no last point, $B$ has last but no first point, and $G$ is good since it is an interval $(a,b)$ where $a$ and $b$ are related by $E$.) Embed $\omega$ cofinally in $X$. Identify each element of $S$ with a color. For points $x$ and $y$ in this embedding, color the set $\{x,y\}$ with the color corresponding to $TH_{\xi}([x,y), P|[x,y))$ (Note that this theory is in $S$ because any two points of $X$ are related). By Ramsey’s theorem [Ra29], there is an infinite homogeneous subset $J$ of points in the embedding, that is, for all $x < y$ in $J$, $\{x,y\}$ has the same color. If $t$ is the $\xi$-theory associated with this color, then $TH_{\xi}(X, P|X) = TH_{\xi}([a,x), P|[a,x)) \oplus (\otimes_{\omega} t)$ for some $x$ in $J$, and is therefore good.

This means that every equivalence class of $E$ is good, and (using the closure of $S$ under addition of theories) that $M/E$ has no jumps.

**Lemma 5.3** If $M$ is a densely ordered countable chain, and $P$ is a sequence of subsets of $M$ whose nonempty members partition $M$, then $\langle M, P \rangle$ has an uniform interval.

**Proof.** We wish to show that there is a uniform interval $\langle I, P|I \rangle$ of $\langle M, P \rangle$. Any interval of $M$ is densely ordered and without endpoints, and is contained in the union of the sets in $P$ restricted to that interval. Hence it is sufficient to show that there is an interval $I$ on which each of the sets in $P|I$ is either empty or is dense in $I$. Proof is by induction on the number of non-empty sets in $P$. If there is only one nonempty set in $P$, then any interval will work. Now assume the lemma is true whenever the number of nonempty sets is less than or equal to $n$, and suppose $P$ contains $n + 1$ nonempty sets. If all of these sets are dense in $M$, then again any interval will work. Otherwise, one of the sets is not dense in $M$ and there is an
interval \((x, y)\) on which this set is empty. Now apply the lemma to \(\langle(x, y), P|(x, y)\rangle\) to obtain \(I\).

If \(\langle M, P \rangle\) is not good, then \(M/E\) has more than one element, and is a densely ordered set. Furthermore (according to Lemma 5.3), it must be the case that \(\langle M/E, E(\xi, P) \rangle\) has a uniform interval \(\langle I/E, E(\xi, P|I) \rangle\). Let \(N\) be some convex subset contained in \([a, b]\) for \(a, b \in I\). Then \(N/E\) consists of two (possibly empty) portions of equivalence classes of \(E\) (which are good), and an interval of \(I/E\). Let \(N'/E\) be this interval. Then since \(\langle N'/E, E(\xi, P|N') \rangle\) is uniform, and the theories of equivalence classes are in \(S\), the \(\xi\)-theory of \(\langle N', P|N' \rangle\) is the shuffle of theories in \(S\), and \(\langle N', P|N' \rangle\), and therefore \(\langle N, P|N \rangle\), is good. Thus any two points of \(I\) are related, which is a contradiction since in this case, \(I\) would be contained in an equivalence class and \(I/E\) could not be an interval of \(M/E\).

**Corollary 5.4** Let \(S\) be the smallest set of \(\xi\) theories that contains the \(\xi\) theory of the singleton chain, and is closed under the operations \(\bigoplus\), \(\bigotimes\), \(\boxtimes\), and \(\sigma\). Then \(S = \text{TH}_{\xi}(K)\).

*Proof.* This follows from Theorem 5.2 and the fact that \(K\) is closed under addition, multiplication by \(\omega\), multiplication by \(\omega^*\), and the shuffle operation on acs.

**Corollary 5.5** Let \(\mathcal{M}\) be the smallest family of chains that contains the singleton chain, \(\{1\}\), and satisfies the following:

1. If \(M_1\) and \(M_2\) are chains in \(\mathcal{M}\), then the chain \(M_1 + M_2\) is in \(\mathcal{M}\).

2. If \(M_1\) is in \(\mathcal{M}\), then so are \(\omega \cdot M_1\) and \(\omega^* \cdot M_1\).

3. If \(N_1, \ldots, N_k\) are in \(\mathcal{M}\), then the shuffle of \(\{N_1, \ldots, N_k\}\) is in \(\mathcal{M}\).

Then for any alternation type \(\xi\), \(\text{TH}_{\xi}(\mathcal{M}) = \text{TH}_{\xi}(K)\).
Proof. The set $\text{TH}_\xi(\mathcal{M})$ is the same as $\mathcal{S}$ of Corollary 5.4.

**Theorem 5.6** Let $\xi$ and $l$ be arbitrary, and let $\{t_1, \ldots, t_n\}$ be a set of $\xi$-theories of acs of length $l$. Then it is possible to compute $\sigma(\{t_1, \ldots, t_n\})$ from $\{t_1, \ldots, t_n\}$.

**Proof.** It is sufficient to show that it is possible to compute $\text{TH}_{T(\xi, l)}(\mathcal{Q}, R)$ where $R = \langle R_t \mid t \in TR_{\xi}(l) \rangle$, $\langle \mathcal{Q}, R \rangle$ is uniform, and $R_t$ is empty if and only if $t \notin \{t_1, \ldots, t_n\}$. Proof is by induction of the blocknumber of $\xi$. Clearly this computation is possible (for any $l$) if $\xi$ is empty, since the 0-theory to be computed depends only on which of the sets in $R$ are empty. (The nonempty sets are pairwise disjoint and dense.) We now consider $\xi = k \cdot \beta$ and assume the computation is possible for all alternation types of blocknumber less than that of $\xi$. It is sufficient (since all subacs of $\langle \mathcal{Q}, R \rangle$ without endpoints have the same $T(\xi, l)$-theory) to compute $S = \{\text{TH}_{T(\beta, l+k)}(X, R|X \bullet P) \}$ where $\langle X, R|X \rangle$ is a subac of $\langle \mathcal{Q}, R \rangle$ and $|P| = |TR_{\beta}(l + k)|$. Note also that the $T(\xi, l)$-theory of a subac of $\langle \mathcal{Q}, R \rangle$ depends only on its 0-theory, since the $T(\xi, l)$-theory of a singleton ac depends only on its 0-theory, and the $T(\xi, l)$-theory of all intervals of $\langle \mathcal{Q}, R \rangle$ is the same. Using this fact, is is easy to see that if $t_1$ and $t_2$ are in $S$, then $t_1 \biguplus t_2$ is in $S$ if and only if it is not the case that $t_1$ is the theory of an ac with a maximum point and $t_2$ is the theory of an ac with a minimum point. Also $\bigotimes_{\omega} t_1$ ($\bigotimes_{\omega^*} t_1$) is in $S$ if and only if $t_1 \bigoplus t_1$ is in $S$. It is possible to compute the set of theories in $S$ which are the theories of singleton acs, since this set depends only on which of the sets in $R$ are empty. Thus, by Theorem 4.1, we would be able to compute $S$ if we knew when the shuffle of theories in $S$ was in $S$, and we could compute the shuffle of any set of theories in $S$. Let $\{s_1, \ldots, s_m\}$ be contained in $S$. Then by the inductive hypothesis, using the fact that the blocknumber of $T(\beta, l + k)$ is the same as that of $\beta$, we can compute $\text{TH}_{T(T(\beta, l+k), [TR_{\xi}(l)] + |TR_{\beta}(l+k)|)}(\mathcal{Q}, R')$ where $\langle \mathcal{Q}, R' \rangle$ is uniform and $R' = \langle R'_t \mid t \in TR_{T(\beta, l+k)}([TR_{\xi}(l)] + |TR_{\beta}(l+k)|) \rangle$ and $R_t$ is empty if and only if $t \notin \{s_1, \ldots, s_m\}$. Thus, it is possible to compute the
shuffle of theories in $S$. Furthermore, whenever \( \{s_1, \ldots, s_m\} \) is a set of theories in $S$, and it is not the case that all are theories of singleton acs and there is a $t \in \{t_1, \ldots, t_n\}$ such that none of the singletons is in $R_t$, then $\sigma(\{s_1, \ldots, s_m\})$ is in $S$. To see this, suppose that $\langle M, R'' \bullet P' \rangle$ is the shuffle of $m$ acs of the form $\langle X, R|X, P \rangle$ each of which is a subac of $\langle Q, R \rangle$ the set of whose theories is $\{s_1, \ldots, s_m\}$. Furthermore assume that for each $t \in \{t_1, \ldots, t_n\}$, some $s_j$ is the theory of an ac whose intersection with $R_t$ is not empty. Here $|R''| = |TR_\xi(l)|$ and $|P'| = |TR_\beta(l + k)|$. Then $M$ is a densely ordered countable chain without endpoints and $R''$ is a partition of $M$ to dense or empty subsets such that for $1 \leq j \leq |TR_\xi(l)|$, $R''_j$ is empty if and only if $R_j$ is empty. In other words, $\langle M, R'' \rangle$ is uniform, and there is an isomorphism $h$ from $M$ to $Q$ such that $h(R''_j) = R_j$ for $1 \leq j \leq |TR_\xi(l)|$. Let $H(P')$ denote the sequence $\langle h(P'_t) \mid t \in TR_\beta(l + k) \rangle$. Since $\langle Q, R \bullet H(P') \rangle \cong \langle M, R'' \bullet P' \rangle$, $TH_{T_\beta(l+k)}(Q, R \bullet h(P')) = TH_{T_\beta(l+k)}(M, R'' \bullet P')$ which is the shuffle of $\{s_1, \ldots, s_m\}$.

**Corollary 5.7** The monadic second-order theory of countable chains is decidable.

**Proof.** We can now compute the set $S$ of Corollary 1 of Theorem 4.1 as follows. We begin with the $\xi$-theory of the singleton chain. At each stage, we add to the set of theories computed so far all theories that can be obtained by applying the operations $\oplus$, $\otimes_\omega$, $\otimes_{\omega^*}$, and $\sigma$ to theories in the set at the previous stage. Because there are a finite number of possible $\xi$-theories, there must be a point at which the set of theories computed at one stage is the same as that at the previous stage. At this point we have computed the desired set $S$.

Since we were interested showing decidability, we stated the proofs in this section in the way we thought most efficient for showing that it was indeed possible to compute the appropriate set of theories for given $\xi$. In doing so, we may have obscured the general nature of the method somewhat. Please note that Theorem
5.2 and its proof can be re-interpreted as a description of the structure of the acs in the class $\mathcal{K}'$. In particular, we have

**Theorem 5.8 (version 2 of Theorem 5.2)** Let $\mathcal{K}'$ be a class of countable acs of length $l$ closed under subacs. Let $\xi$ be an arbitrary alternation type. Let $\mathcal{M}'$ be a family of chains that contains each singleton ac in $\mathcal{K}'$, and satisfies the following:

1. If $M_1$ and $M_2$ are acs in $\mathcal{M}'$, then the ac $M_1 + M_2$ is in $\mathcal{M}'$.

2. $\mathcal{M}'$ contains the sum of acs $\widetilde{M}_i$ for $i \in \omega^{(*)}$ whenever each ac $\widetilde{M}_i$ is in $\mathcal{M}'$, and for all $i \in \omega^{(*)}$, $TH_\xi(\widetilde{M}_i) = t$ for some $t \in TR_\xi(l)$.

3. $\mathcal{M}'$ contains the sum of acs $\widetilde{M}_i$ for $i \in Q$ whenever each ac $\widetilde{M}_i$ is in $\mathcal{M}'$, and the sets $E_t = \{i \in Q \mid TH_\xi(\widetilde{M}_i) = t\}$ for $t \in TR_\xi(l)$ form a partition of $Q$ to dense or empty subsets.

Then every ac in $\mathcal{K}'$ is in (is isomorphic to) an ac in $\mathcal{M}'$. 
6. A General Template

In section 3 we gave a description of the composition method that applied in
the applications of that section, in which we combine two or three models for a
language to form a new model. In our later discussion of chains, the composition
often involves an infinite number of models. Still there is the general theme of
compositions of models used to build a certain class of models, and operations on
theories that correspond to these compositions. In this section, we give a more
general template for composition theorems. This template is based on those
found in [Sh75] and [Gu79].

First we describe a special kind of composition of models. We consider only
models for formally first order languages in which the variables represent sets, that is
for monadic languages as discussed in section 3. We begin with two classes of models
\( \mathcal{M} \) and \( \mathcal{I} \) for languages \( L_\mathcal{M} \) and \( L_\mathcal{I} \) respectively. We assume both languages extend
the language \( BOOL \) (of section 4) which is the first-order language of boolean
algebras containing all the usual boolean operations, constants 0 and 1, and the
equality predicate. We also assume that in addition to the vocabulary of \( BOOL \),
both languages contain only finitely many predicates, and finitely many constants.
Furthermore, we assume that all of the models in \( \mathcal{M} \) interpret these constants as sets
of cardinality less than or equal to one, that is, they are either empty or singletons.
We do this partly because it is a natural situation (as in the applications), and
in order to simplify the exposition. It is certainly possible to include constants
that represents arbitrary sets (besides 0 and 1 of \( BOOL \)), and we discuss this later.
Elements of \( \mathcal{I} \) are intended as index structures for compositions of models. We refer
to a set of the form \( \{M_i\}_{i \in I} \) where each \( M_i \) is a \( L_\mathcal{M} \) model, the \( M_i \)'s are pairwise
disjoint, and \( I \) is an an \( L_\mathcal{I} \) model as an \( I \) family of \( \mathcal{M} \) models. We construct new
models for \( L_\mathcal{M} \) from such families as follows. A composition rule \( \sigma \) is an operation
that assigns to certain \( \mathcal{I} \) families of \( \mathcal{M} \) models a model for \( L_\mathcal{M} \) as follows. Let
\( F \) be such a family, that is \( F = \{M_i\}_{i \in I} \). For \( t \in PT^0(0) \), we let
\( E_t(F, 0, \lambda) = \)
\{i \in I \mid TH^0(M_i) = t\}, and we let \(E(F,0,\lambda) = \langle E_t(F,0,\lambda) \mid t \in PT^0(0)\rangle\) where the elements of \(PT^0(0)\) are in their standard (lexicographical) order. With the rule \(\sigma\) is associated a quantifier-free formula of \(L_I\), \(\sigma_d\), with free variables among \(\{v_t \mid t \in PT^0(0)\}\). In order to simplify the exposition, we think of the index \(t\) here (and in analogous situations involving \(PT_\xi(l)\) for arbitrary \(\xi\) and \(l\)) simultaneously as an element of \(PT^0(0)\) and as the number of that element in the standard ordering. Then \(\sigma_d(E(F,0,\lambda))\) denotes the value of the formula \(\sigma_d\) when the set \(E_t(F,0,\lambda)\) is substituted for variable \(v_t\). The composition according to the rule \(\sigma\) is defined if and only if \(\sigma_d(E(F,0,\lambda))\) holds in \(I\). The rule \(\sigma\) assigns to each constant \(c\) of \(L_M\) predicates \(R_c\), \(EQ_c\) and \(G_c\) such that for each \(I \in \mathcal{I}\), \(G_c\) is an equivalence relation on individual elements of \(I\), for sets \(X\) and \(Y\) of \(I\), \(R_c(X,Y)\) if and only if there is an \(i \in X\) and there is a \(j \in Y\) such that \(G_c(i,j)\), and for set \(X\) in \(I\), \(EQ_c(X)\) if \(X\) is not empty and all elements of \(X\) are related by \(G_c\). Several constants may be assigned the same \(G_c\). In the simplest case \(G_c\) can be interpreted as equality, in which case \(R_c(X,Y)\) means that \(X \cap Y\) is not empty, and \(EQ_c(X)\) means that \(X\) is a singleton. The predicates of the form \(G_c\) will be used to identify (i.e. glue as in the second example of section 3) individual elements from the family of original models in the new model. If each \(G_c\) is merely equality, no individual elements are to be glued. We denote by \(c_i\), the interpretation of the constant \(c\) in \(M_i\). We define an equivalence relation on \(\bigcup_{i \in I} M_i\) as follows. Two elements \(x\) and \(y\) of this union are related by this relation if and only if \(x = y\) or for some \(c\) in \(L_M\), \(x \in c_i\) and \(y \in c_j\) and \(G_c(i,j)\). If \(C = \sigma(F)\) then \(|C|\) is the set of equivalence classes determined by this relation on \(\bigcup_{i \in I} M_i\). We write \([x]\) to denote the equivalence class of \(x\). If \(U\) is subset of \(|C|\), we write \(U|M_i\) to denote \(\{x \in M_i \mid [x] \in U\}\). Note that if no gluing is done, this set is merely \(U \cap M_i\). If \(U = \langle U_1, \ldots, U_l\rangle\) is a sequence of subsets of \(|C|\) and \(\xi\) is any alternation type, we let \(E_t(F,\xi,U) = \{i \in I \mid TH_\xi(M_i,U_1|M_i,\ldots,U_l|M_i) = t\}\), and we let \(E(F,\xi,U) = \langle E_t(F,\xi,U) \mid t \in PT_\xi(k)\rangle\). (This is essentially the sequence \(E(\xi,U)\) of section 4.) Informally, we refer to the sets \(E_t(F,\xi,U)\) as \((\xi,l)\)-index sets. Subsets of \(I\) of the form \(E(F,\lambda,U)\) (also denoted by \(E(F,0,U)\)) are used to define
the truth value of atomic formulas in the new model. To each predicate \( R \) (other than equality) of \( L_M \), \( \sigma \) assigns a quantifier-free formula \( \sigma_R \) of \( L_I \) in variables \( v_t \) for \( t \in \PT^0(k) \) where \( k \) is the arity of \( R \). This formula defines \( R \) in the new model \( C = \sigma(F) \). In particular, for subsets \( U_1, \ldots, U_l \) of \( |C| \), \( R(U_1, \ldots, U_l) \) holds if and only if \( I \models \sigma_R(E(F,0,U)) \), which we write in this section also as \( \langle I, E(F,0,U) \rangle \models \sigma_R \). The rule \( \sigma \) must also specify the interpretation of the constants other than 0 and 1 of \( BOOL \) (we assume these have their usual meaning in all models). For each constant \( c \) of \( L_M \) and \( i \in I \), \( c|M_i \) is defined by the rule \( \sigma \) either to be \( c_i \), or to have the value 0. For each such \( c \), \( I \) must be split into two parts such that whenever \( i \) is in one these parts, \( c|M_i \) is interpreted as the emptyset, and for the remaining values of \( i \), \( c|M_i \) is to be \( c_i \). For each constant \( c \) of \( L_M \), let \( FULL_c(F) \) be the portion of \( I \) on which \( c|M_i \) is interpreted as \( c_i \). The rule \( \sigma \) specifies this set as the value of a term \( t_c \) involving variables from the set \( \{v_t \mid t \in \PT^0(0)\} \) and the constants of \( L_I \) when \( E_t(F,0,0) \) is substituted for \( v_t \) for each \( t \in \PT^0(0) \). We denote this value (and we denote similarly the value of a term evaluated at a tuple of sets indexed by theories in their standard order) as \( t_c(E(F,0,0)) \). We assume that \( \sigma_d \) requires that the interpretation of distinct constants in each \( M_i \) are disjoint sets, that for all constants \( c \) of \( L_M \) the set \( t_c(E(F,0,0)) \) and its complement are not related by \( R_c \), that for each \( c \), either \( t_c(E(F,0,0)) \) is empty or \( EQ_c(t_c(E(F,0,0))) \) holds (These last two requirements say that \( t_c(E(F,0,0)) \) is empty or is an equivalence class of \( G_c \). The second one is necessary only if we wish to make sure that \( c \) in the new model is a singleton), and that whenever two 0,0- index sets \( X \) and \( Y \) are related by a predicate \( R_c \), then \( TH^0(M_i) \) contains \( c = 0 \) if and only if \( TH^0(M_j) \) does for any \( i \in X \) and \( j \in Y \). We may strengthen this last requirement as was done in the application for graphs of Section 3 to require that the portions of models being identified are isomorphic (but this is not necessary to the proof). In particular, we may require that if \( S \) is the set of constants \( c \) such that two 0,0-index sets \( X \) and \( Y \) are related by \( R_c \), then \( TH^0(M_i) \) and \( TH^0(M_j) \) agree on all formulas containing only constants from \( S \) in addition to those from \( BOOL \) for any \( i \in X \) and \( j \in Y \).
The template can easily be extended to include constants that represent arbitrary (not only empty or singleton) sets, but it is not natural to view them as being shared or identified. Such constants are not associated with predicates of the form $G_c$ used in identifying individual elements, however they must be assigned a term of the form $t_c$ in the same manner as above. Also the formula $\sigma_d$ must be strengthened to ensure certain compatibility conditions (i.e that for constant $d$ as interpreted in the new model, $d|M_i$ which is by definition $d_i$ for $i \in t_d(E(F,0,0))$ and 0 otherwise is also equal to $\{x \in M_i \mid [x] \in d\}$.

We extend the notion of a family of models and our composition operation to one on augmented models as follows. Let $s$ and $t$ be elements of $PT_\xi(l)$ for some $\xi$ and $l$. We say $s$ and $t$ are $c$-compatible if for $1 \leq j \leq l$, $c \cap v_j = c$ is in the $l$-trace of one if and only if it is in the $l$-trace of the other. Let $t^c_{\xi,l}$ be the term obtained by substituting the union of all variables $v_t$ with $t \in PT_\xi(l)$ such that the trace of length $0$ of $t$ is $s$, in place of $v_s$ in $t_c$. Let $\sigma^c_{\xi,l}$ be the formula obtained from $\sigma_d$ by the same substitution. Suppose $\mathcal{M}$ is a class of augmented models of weight $l$ for $L_\mathcal{M}$. Suppose $F_a = \{\langle M_i, P_i \rangle \}_{i \in I}$ where for $i \in I$, $\langle M_i, P_i \rangle$ is in $\mathcal{M}$, and $P_i = \langle P_{i1}, \ldots, P_{il} \rangle$. We still refer to $F_a$ as an $\mathcal{I}$ family of $\mathcal{M}$ models. Let $F = \{M_i\}_{i \in I}$. We let $\sigma^c_{\xi,l}$ be the conjunction of $\sigma^c_{\xi,l}$ and the quantifier-free formula (also with free variables in the set $\{v_t \mid t \in PT_\xi(l)\}$) that requires for each constant $c$, that $R_c(v_t, v_s)$ implies that $s$ and $t$ are $c$-compatible theories. Let $E'(F_a, \xi) = \langle E'_t(F_a, \xi) \mid t \in PT_\xi(l) \rangle$ where for each such $t$, $E'_t(F_a, \xi) = \{i \in I \mid TH_\xi(M_i, P_i) = t\}$. Then, whenever $\sigma^\lambda_{\xi,l}(E'(F_a, \lambda))$ holds, we define $\sigma(F_a)$ to be the augmented model $\langle \sigma(F), U_1, U_2, \ldots, U_l \rangle$ where for $1 \leq j \leq l$, $U_j = \{[x], x \in \cup_{i \in I} P_{ij}\}$. It is important to note that $\sigma^\xi_{\xi,l}(E'(F_a, \xi))$ holds in $I$ if and only if $\sigma^\lambda_{\xi,l}(E'(F_a, \lambda))$ does. More importantly, whenever $\sigma^\xi_{\xi,l}(E'(F_a, \xi))$ does hold, then $\sigma(F_a)$ is defined and $E'(F_a, \xi)$ is the same sequence as $E(F, \xi, U)$. In fact, $\sigma^\xi_{\xi,l}$ is specifically designed to ensure this, in particular, to require that for each $i \in I$ and $j$ between 1 and $l$, $U_j|M_i = P_{ij}$ and therefore $TH_\xi(M_i, P_i) = TH_\xi(M_i, U|M_i)$. Finally, note that $t_c(E(F,0,0)) = t^c_{\xi,l}(E(F, \xi, U))$. 
Theorem 6.1 Let $\mathcal{M}$ be a class of augmented models for $L_{\mathcal{M}}$ of weight $l$. Let $F_a = \{ (M_i, P_i) \}_{i \in I}$ be an $I$ family of $\mathcal{M}$ models, and let $F$ denote $\{ M_i \}_{i \in I}$. Suppose that $\sigma(F_a)$ is defined as above and equals $\langle \sigma(F), U \rangle$. Then for all alternation types $\xi$, $\text{TH}_\xi(\sigma(F_a))$ is computable from $\text{TH}_{T(\xi, l)}(I, E(F, \xi, U))$.

Proof. As usual, we begin by showing the claim for empty $\xi$ and proceed by induction on the blocknumber of $\xi$. Let $\tau$ be a term in $L_{\mathcal{M}}$, and for each $i \in I$ let $\tau^i$ be the term obtained from $\tau$ by substituting 0 for each constant $c$ such that $i$ that is not in $t^l \cdot (E(F, 0, U))$. It is easy to see that $\tau(U|M_i) = \tau^i(U|M_i) = \tau^i(P_i)$. Also, if $\delta$ is another such term, then $\delta(U) = \tau(U)$ if and only if $\delta^i(U|M_i) = \tau^i(U|M_i)$ for each $i$. This means that it is possible to determine equality in the new model from $\text{TH}^0(I, E(F, 0, U))$. Now suppose that $\tau_1, \ldots, \tau_k$ are all terms in variables $v_1, \ldots, v_l$. In order to establish the claim for empty $\xi$ it is sufficient to show that for all such sequences of terms, $\text{TH}^0(I, E(F, 0, \tau_1(U), \tau_2(U), \ldots, \tau_k(U))$ is computable from $\text{TH}^0(I, E(F, 0, U))$. But this is possible since for each $i \in I$, $\text{TH}^0(M_i, \tau_1(U)|M_i, \ldots, \tau_k(U)|M_i))$ is equal to $\text{TH}^0(M_i, \tau_1^i(U|M_i), \ldots, \tau_k^i(U|M_i))$, and this theory is determined by $\text{TH}^0(M_i, U|M_i)$ and the membership of $i$ in each of the terms $t^l \cdot (E(F, 0, U))$ where $c$ is a constant of $L_{\mathcal{M}}$. (In particular, the set of $i \in I$ such that $\text{TH}^0(M_i, \tau_1^i(U|M_i), \ldots, \tau_k^i(U|M_i))$ equals $r$ for a given $r \in PT^0(k)$ is a union of sets of the form $E_{t}(F, 0, U) \cap C$ where $C$ is an intersection of sets each of which equals $t^l \cdot (E(F, 0, U))$ or its complement.)

Now assume that $\xi = k \cdot \beta$. We wish to compute $\text{TH}_\xi(\sigma(F_a))$ which is equal to the set

$$S = \{ \text{TH}_\beta(\sigma(F), U \bullet Q) \mid |Q| = k \}.$$ 

By the inductive hypothesis $S$ is computable from

$$S_1 = \{ \text{TH}_{T(\beta, l+k)}(I, E(F, \beta, U \bullet Q)) \mid |Q| = k \}.$$ 

Recall that $T(\xi, l) = |PT_\beta(l+k)| \cdot T(\beta, l+k)$. The set $S_1$ is computable from

$$S_2 = \{ \text{TH}_{T(\beta, l+k)}(I, E(F, \xi, U \bullet E(F, \beta, U \bullet Q)) \mid |Q| = k \}.$$
But $S_2$ is computable from

$$TH_{T(\xi, l)}(I, E(F, \xi, U)) = \{TH_{T(\beta, l+k)}(I, E(F, \xi, U) \cdot R) \mid |R| = |PT_\beta(l+k)|\}$$

since any element in this set such that the nonempty sets of $R$ form a partition of $I$, and such that $t$ is an element of $s$ whenever $R_t \cap E_s(F, \xi, U)$ is not empty, and such that $\sigma_d(\xi, l+k)(R)$ holds in $I$ is in $S_2$. ■

For a class of models $\mathcal{K}$, we denote the class of augmented models of the form $(K, P)$ with $K \in \mathcal{K}$ of weight $l$ by $\mathcal{K}(l)$. From now on, we speak of augmented models and regard “non-augmented” models as a special case, that is as augmented models of weight 0. In the discussion of closure that follows, we allow families of models from a class also to contain isomorphic copies of models in the class. Now suppose that $B$ is a class of augmented models for $L_M$ of weight $l$, and $I$ is a class of models for $L_I$ as above. Let $C$ be the class of models of the form $\sigma(F)$ where $F$ is an $I$ family of $B$ models and $\sigma(F)$ is defined. Then we say that $C = COMP(\sigma, B, I)$. Suppose that $M$ is the smallest class of augmented $L_M$ models that contains $B$ and contains $\sigma(F)$ whenever $F$ is an $I$ family of $M$ models (and $\sigma(F)$ is defined). Then we say that $M = CLOSURE(\sigma, B, I)$. Then we have the following two theorems.

**Theorem 6.2** Suppose that $C = COMP(\sigma, B, I)$. Then for any alternation type $\xi$, $TH_{\xi}(C)$ is computable from $TH_{\xi}(B)$ and $TH_\eta(I(PT_{\xi}(l)))$ where $\eta = T(\xi, l)$.

**Theorem 6.3** Suppose that $M = CLOSURE(\sigma, B, I)$. Then for any alternation type $\xi$, $TH_{\xi}(M)$ is computable from $TH_{\xi}(B)$ and $TH_\eta(I(PT_{\xi}(l)))$ where $\eta = T(\xi, l)$.

Thus, for example, it is possible to show the decidability of the theory of $TH(M)$ by showing that $M$ is equal to $COMP(\sigma, B, I)$ or to $CLOSURE(\sigma, B, I)$, where the classes $B$ and $I$ are “simpler” in the sense that it is known that for each $\xi$, $TH_{\xi}(B)$ and $TH_\eta(I(PT_{\xi}(l)))$ are computable. For example, it may be the case that the only index structure used is a finite chain, in which case the theories involving the index structures are clearly computable.
The only clear way to express a class of augmented models \( \mathcal{M} \) as a composition or closure may, however, involve index structures which seem to be no less complicated than elements of \( \mathcal{M} \) themselves. It may still be possible to compute the necessary theories by placing conditions, not on the elements of \( \mathcal{I} \) themselves, but rather on the index sets in these elements that may appear in applying the rule \( \sigma \). For a moment let us restrict our attention to a fixed alternation type \( \xi \). Suppose that for this \( \xi \), there is a formula (not necessarily quantifier-free) \( \psi \) of \( L_\mathcal{I} \) in variables \( v_t \) for \( t \in PT_{\xi}(l) \) (where \( l \) is the weight of elements of \( \mathcal{M} \)) such that \( \mathcal{M} \) is the smallest class of augmented models that contains \( \mathcal{B} \) and contains \( (\mathcal{F}_A) \) whenever \( \mathcal{F}_A = \{ h_{M_i}; P_i \mid i \in I \} \) with each \( (M_i, P_i) \in \mathcal{M} \), and \( \psi \land \sigma_{\xi l}^\mathcal{A}(E'(\{M_i\}_{i \in I}, \xi)) \) holds in \( I \). Then we say that \( \mathcal{M} = \psi - CLOSURE(\sigma, \mathcal{B}, \mathcal{I}(PT_{\xi}(l))) \). We refer to \( \psi \) as a condition formula. We define the \( \psi - COMP(\sigma, \mathcal{B}, \mathcal{I}(PT_{\xi}(l))) \) analogously. Then we have:

**Theorem 6.4** Suppose \( \mathcal{C} = \psi - COMP(\sigma, \mathcal{B}, \mathcal{I}(PT_{\xi}(l))) \). Then we can compute \( TH_{\xi}(\mathcal{C}) \) from \( TH_{\xi}(\mathcal{B}) \) and \( \{ TH_{T(\xi,I)}(\tilde{I}) \mid \tilde{I} \in \mathcal{I}(PT_{\xi}(l)) \text{ and } \tilde{I} \models \psi \} \).

**Theorem 6.5** Suppose \( \mathcal{M} = \psi - CLOSURE(\sigma, \mathcal{B}, \mathcal{I}(PT_{\xi}(l))) \). Then we can compute \( TH_{\xi}(\mathcal{M}) \) from \( TH_{\xi}(\mathcal{B}) \) and \( \{ TH_{T(\xi,I)}(\tilde{I}) \mid \tilde{I} \in \mathcal{I}(PT_{\xi}(l)) \text{ and } \tilde{I} \models \psi \} \).

**Proof.** We can begin with the set that is \( TH_{\xi}(\mathcal{B}) \). At successive stages, we add to our set of theories the prototype of all elements \( TH_{T(\xi,I)}(I, R) \) in \( \{ TH_{T(\xi,I)}(\tilde{I}) \mid \tilde{I} \in \mathcal{I}(PT_{\xi}(l)) \text{ and } \tilde{I} \models \psi \} \) such that the nonempty sets in \( R \) partition \( I \), the set of indices of the nonempty sets in \( R \) is contained in the set created at the previous stage, and \( \sigma_{\xi l}^\mathcal{A}(R) \) holds in \( I \). Eventually no new theories will be added and at this point we have computed the set \( TH_{\xi}(\mathcal{M}) \).

In the examples of section three, \( \mathcal{I} \) consisted of a single finite chain, and all the formulas placing conditions on when a sum was included (all formulas of the form \( \psi \) above) were equivalent to conditions on the \( \xi \) theories of the models or augmented models to be combined. For the class of countable linear ordered sets, there were actually several classes \( \mathcal{I} \) (The definition above on closure and the result
on computability of theories can be extended to cover this case in the obvious way.),
one consisting of the two point chain, one consisting of \( \omega \), one consisting of \( \omega^* \), and
one consisting of \( \mathcal{Q} \). For sums indexed by \( \omega(\omega^*) \) and given alternation type \( \xi \),
the condition formula merely required that all the acs in the sum have the same
\( \xi \)-theory. For sums indexed by the rationals, the condition formula required the
uniformity of acs of the form \( \langle \mathcal{Q}, R \rangle \).

In cases in which the index structures are no less complicated than the original
class, the condition formulas are used to allow a form of bootstrapping to be done in
which theories of a smaller blocknumber can be used to calculate the set of theories
that meet the condition formula. Finally we have

**Theorem 6.6** Suppose that for each alternation type \( \xi \) and weight \( l \) there is a
formula \( \psi_{(\xi,l)} \) such that \( \mathcal{M}(l) = \psi_{\xi,l} - CLOSURE(\sigma, \mathcal{B}(l), \mathcal{T}(PT_{\xi}(l))) \) [or \( \mathcal{M}(l) = \psi_{\xi,l} - COMP(\sigma, \mathcal{B}(l), \mathcal{T}(PT_{\xi}(l))) \)], \( TH_\xi(\mathcal{B}(l)) \) is computable, and for any \( n \), the set
\( \{TH^0(\vec{I}) \mid \vec{I} \in \mathcal{T}(PT^n(n)), \vec{I} \models \psi_{(0,n)} \} \) is computable. Suppose also that for each alternation type \( \zeta \) and \( m \) such that \( \zeta = k \cdot \tau \), \( \{TH_{T(\zeta,m)}(\vec{I}) \mid \vec{I} \in \mathcal{T}(PT_{\zeta}(m)) \) and
\( \vec{I} \models \psi_{(\zeta,m)} \} \) is computable from \( \{TH_{T(\eta,j)}(\vec{I}) \mid \vec{I} \in \mathcal{T}(PT_{\eta}(j)), \vec{I} \models \psi_{(\eta,j)} \} \},
where \( \eta = T(\tau, m + k) \), and \( j = |PT_{\zeta}(m)| + |PT_{\tau}(m + k)| \). Then \( TH_\xi(\mathcal{M}(l)) \) is
computable for any \( \xi \) and \( l \).

**Proof.** By theorem 6.5 [ or 6.4], \( TH_\xi(\mathcal{M}(l)) \) is computable from \( TH_\xi(\mathcal{B}(l)) \) and the set \( \{TH_{T(\xi,l)}(\vec{I}) \mid \vec{I} \in \mathcal{T}(PT_{\xi}(l)) \) and \( \vec{I} \models \psi_{(\xi,l)} \} \}. It is sufficient then to show
that for any \( \xi \) and \( l \), the second set is computable. This is true for empty \( \xi \) and any
weight \( l \) by assumption. Assume it is true for any alternation type of blocknumber
less than that of alternation type \( \xi = k \cdot \beta \), and any weight \( j \). We wish to show that
the assumption implies that the set is computable for \( \xi \) and any weight \( l \). But this
is clear since \( \{TH_{T(\eta,j)}(\vec{I}) \mid \vec{I} \in \mathcal{T}(PT_{\eta}(j)), \vec{I} \models \psi_{(\eta,j)} \} \) where \( \eta = T(\beta, l + k) \),
and \( j = |PT_{\xi}(l)| + |PT_{\beta}(l + k)| \) is computable since the blocknumber of \( \eta \) is the
same as that of \( \beta \) which is less than that of \( \xi \).
7. A Final Application

In this section we give a last application, illustrating the use of Theorem 6.1. This example is the extension of the result on nostrums of width \( k \) from section 3 to the case of what we call infinite binary nostrums. We use the decidability of \( S2S \) [Ra69] to obtain our decidability result. Essentially, the only index structure used in forming our compositions is the infinite binary tree \( T \) of section 3. First, we define an infinite binary nostrum. An infinite binary nostrum consists of a graph \( N_G \) along with a function that assigns to each node \( t \) of the infinite binary tree \( T \) a subset (possibly empty) of the vertices of \( N_G \) called the grasp of \( t \) such that

1. each vertex of \( N_G \) is in the grasp of some node of \( T \),

2. every edge of \( N_G \) has both of its ends in the grasp of some node, and

3. for each vertex \( v \), the set of nodes of \( T \) that grasp \( v \) is connected (when \( T \) is viewed as an undirected graph).

The width of the nostrum is the maximum cardinality of the grasp of any node \( T \).

Nostrums of width \( k \) are models for the language described below. We denote by \( |N| \), the union of the vertex set of \( N_G \), the edge set of \( N_G \) and the node set of \( T \). Our monadic second-order language for these nostrums, \( L^k_T \) is a slight modification of the language \( L^k_N \) of section two. The language \( L^k_T \) contains in addition to \( BOOL \) the binary predicates \( INC \), \( LC \), \( RC \) and \( GRASPS \), along with unary predicates \( SING \), \( VERT \), and \( EDGE \), and \( NODE \). If \( X \) and \( Y \) are set variables representing subsets of \( |N| \), \( SING(X) \) means \( X \) has exactly one member, \( VERT(X) \) means that all members of \( X \) are vertices, \( EDGE(X) \) means that all members of \( X \) are edges, and \( NODE(X) \) means that all members of \( X \) are nodes. We use \( INC(X,Y) \) to
mean that $X$ and $Y$ are singleton sets, $X$ contains a single edge, $Y$ contains a single vertex, and the edge in $X$ is incident to the vertex in $Y$. $LC(X,Y)$ ($RC(X,Y)$) means that $X$ and $Y$ are sets consisting of a single node, and that the node in $Y$ is the left (right) child of the node in $X$. We use the predicate $GRASPS(X,Y)$ to denote that $X$ is a set consisting of a single node, $Y$ is a set consisting of a single vertex, and the vertex in $Y$ is in the grasp of the node in $X$. Finally our language contains for and $1 \leq i < j \leq k$, constants $c_i$, and $e_{ij}$. Note that $B_k$ of section 2, the class of ordinary $k$-nostrums whose tree part consists of a single node (and $\bigcup_{1 \leq i \leq k} c_i$ contains all the vertices of $B_g$ for $B \in B$) is also a class of models for $L^k_T$ and that $TH(B_k)$ is decidable. The language $L^k_T$ plays the role of $L_M$ of section 6. The language $L_T$ contains $BOOL$, the predicates $LCHILD(X,Y)$, $RCHILD(X,Y)$, $SING(X)$, and (for each $i$ and $j$ between 1 and $k$) $L_T$ contains predicates $R_i(X,Y)$, $G_i(X,Y)$, $EQ_i(X)$, $R_{ij}(X,Y)$, $G_{ij}(X,Y)$, and $EQ_{ij}(X)$. Finally $L_T$ contains a constant $C_S$ for each nonempty subset $S$ of $\{1, \ldots, k\}$. The class $I$ of models for $L_T$ consists of copies of the infinite binary tree $T$ in which the constants $C_s$ are disjoint subsets, and in which the meanings of the predicates $G_i$ and $G_{ij}$ are as described below (and the remaining predicates have the meanings associated with these as described in section 6). For $1 \leq i \leq l$, $G_i$ is the transitive closure of the relation $\sim_i$ where for $t, r \in T$, $r \sim_i t$ if and only if $t = r$, or $LCHILD(t,r)$ and $r \in C_s$ and $i \in S$, or $RCHILD(t,r)$ and $r \in C_s$ and $i \in S$, or $LCHILD(r,t)$ and $t \in C_s$ and $i \in S$, or $RCHILD(r,t)$ and $t \in C_s$ and $i \in S$. For $1 \leq i, j \leq k$ and sets $X$ and $Y$, $G_{ij}(X,Y)$ holds if both $G_i(X,Y)$ and $G_j(X,Y)$ do. Intuitively, each model in $I$ is a partially labeled infinite binary tree. The label of a vertex $t$ is $S$ whenever $t \in C_S$, and in any composition of an $I$ family of $B$ structures, the intended meaning of $t \in S$ is that $B_t$ shares the vertices $c_n$ for $n \in S$ with $B_j$ where $j$ is the parent of $t$. We assume that the formula $\sigma_d$ also requires that whenever $t \in C_S$, the constant $c_n$ is not empty in $B_t$ for $n \in S$. 
Lemma 7.1 Let $\mathcal{I}$ be the class of models for $L_{\mathcal{I}}$ consisting of copies of the infinite binary tree in which the constants $C_s$ represent disjoint sets of nodes of the tree and the the predicates $G_i$, $G_{ij}$, $E_i$, $E_{ij}$, $R_i$, and $R_{ij}$ have the meaning described above. Then for any alternation type $\xi$, $TH_\xi(\mathcal{I})$ is computable.

Proof. The decidability of $TH(\mathcal{I})$ follows from the decidability of S2S. Note that once the constants have been added to the language of S2S, all of the predicates and operations of BOOL are definable in S2S. In particular, the transitive closure of a relation is definable in monadic second-order logic.

Let $N$ be an infinite binary nostrum of width $k$. It is easy to see that the vertices of $N_G$ can be colored with $k$ colors such that no two vertices in the grasp of the same node have the same color. Consider the model $I$ of $\mathcal{I}$ (as described in the Lemma above) in which a node $i$ of the infinite binary tree $T$ is in the set $C_S$ if $S$ is the nonempty set of elements $j$ of $\{1, \ldots, k\}$ such that there is a vertex in the grasp of both $i$ and its parent that has color $j$. Also for node $i$, let $B_i$ be the model for $L^k_{T_i}$ whose graph is the subgraph of $N_G$ induced by the vertices in the grasp of $i$, and in which for $1 \leq j \leq k$ the constant $c_j$ is the vertex in this grasp colored with the $j$th color if there is one and is empty otherwise. Then it is easy to see that $N$ is $\sigma(\{B_i\}_{i \in I})$ where $\sigma$ is the composition rule defined as follows. The rule $\sigma$ assigns to each constant $c_i$ the predicate $G_i$, and to each constant $e_{ij}$ the predicate $G_{ij}$ for the purposes of identifying vertices and edges in the grasp of several vertices. All constants of the form $c_i$ and $e_{ij}$ in the new model are interpreted as the emptyset (i.e. the term associated with each such constant is the emptyset). Intuitively, the formula $\sigma_d$ (as described in the last section) ensures that the portions of the graph being identified in the composition are isomorphic. It remains to see that it is possible to define formulas $\sigma_R$ for each predicate $R$ of $L^k_T$.

The formulas we discuss will contain variables of the form $v_t$ where $t$ is a possible
0-theory, that is $t \in PT^0(l)$ for some $l$. For the purposes of discussion, we think of the variables appearing in formulas of $t$ as $x_1, \ldots, x_l$ in order to distinguish them from the variables $v_t$. The formulas $\sigma_{\text{VERT}}$, $\sigma_{\text{EDGE}}$, and $\sigma_{\text{NODE}}$ are quantifier-free formulas in variables $v_t$ for $t \in PT^0(1)$ where each $v_t$ is intended to represent a possible 0,1-index set. The formula $\sigma_{\text{VERT}}$, for example, says that $v_t$ equals 0 unless $t$ is a theory that contains $\text{VERT}(x_1)$. Formulas $\sigma_{\text{EDGE}}$ and $\sigma_{\text{NODE}}$ are defined similarly. The formula $\sigma_{\text{SING}}$ is again more complicated because of the gluing of vertices. The formula $\sigma_{\text{SING}}$ is the disjunction of all formulas of the form $(\bigwedge_{t \in PT^0(1)} \psi_t) \land SING(v_j)$ where each $\psi_i$ is either $v_i = 0$ or $\neg(v_i = 0)$, $\psi_j$ says that $v_j$ is not 0, $v_j$ contains $SING(x_1)$, and for all other $i$ such that $\psi_i$ says that $v_i$ is not empty, $i$ is a theory that contains $x_1 = 0$, along with the disjunction of all formulas of the form $(\bigwedge_{t \in PT^0(1)} \psi_t) \land EQ_c(\bigcup_{j \in Q} v_q)$ where the formulas $\psi_i$ are as before, $Q$ is the set of $i$ such that $\psi_i$ says that $v_i$ is not empty and $i$ is a theory that does not contain $x_1 = 0$, and it is also the case that every $i$ in $Q$ contains $x_1 = c$. (Here $c$ is any of the constants $c_i$ or $e_{ij}$.) The formula $\sigma_{\text{LC}}$ has variables $v_t$ for $t \in PT^0(2)$ and asserts that for some $t_1$ and $t_2$, all $v_t$ for $t \neq t_1$ are either empty or contain $x_1 = 0$, all $v_t$ for $t \neq t_2$ are either empty or contain $x_2 = 0$, $v_{t_1}$ and $v_{t_2}$ are singletons, $\text{LCHILD}(v_{t_1}, v_{t_2})$ holds, $t_1$ contains $\text{NODE}(x_1)$, and $t_2$ contains $\text{NODE}(x_2)$. The formula $\sigma_{\text{RC}}$ is defined similarly. The formula $\sigma_{\text{GRASPS}}$ also contains variables $v_t$ for $t \in PT^0(2)$. We wish $\sigma_{\text{GRASPS}}(E(F, 0, X, Y))$ to hold whenever $X$ is a singleton node, $Y$ is a singleton vertex and $Y$ is in the grasp of $X$ in the resulting nostrum.

We can construct formulas similar to $\sigma_{\text{SING}}$, $\sigma_{\text{NODE}}$, and $\sigma_{\text{VERT}}$ to assert that $X$ is a singleton node, and $Y$ is a singleton vertex. In order to form $\sigma_{\text{GRASPS}}$ we take the conjunction of these formulas and a formula that says that there is some $t$ that contains $\text{GRASPS}(x_1, x_2)$ such that $v_t$ is not empty. The formula $\sigma_{\text{INC}}$ also contains variables $v_t$ for $t \in PT^0(2)$. We wish $\sigma_{\text{INC}}(E(F, 0, X, Y))$ to hold whenever $X$ is a singleton edge, $Y$ is a singleton vertex and $X$ is incident to $Y$ in the graph part of resulting nostrum. We can construct formulas similar to $\sigma_{\text{SING}}$,
$\sigma_{\text{EDGE}}$, and, and $\sigma_{\text{VERT}}$ to assert that $X$ is a singleton edge, and $Y$ is a singleton vertex. In order to form $\sigma_{\text{INC}}$ we take the conjunction of these formulas and a formula that says that there is some $t$ that contains $\text{INC}(x_1, x_2)$ such that $v_t$ is not empty.

Clearly $\sigma$ applied to an $\mathcal{I}$ family of $B_k$ structures produces an infinite binary nostrum of width $k$ in which vertices in the grasp of node $i$ are precisely those in the element of $B_k$ associated with $i \in I$. Thus the class of infinite binary nostrums of width $k$ equals $\text{COMB}(\sigma, B_k, \mathcal{I})$ and is decidable by Theorem 6.2. The notion of a tree-decomposition is extended to infinite graphs and infinite trees in the obvious manner. Any infinite graph that has a tree-decomposition of finite width in which the tree is countable has a tree-decomposition of the same width in which the tree is the infinite binary tree [Co92]. Actually, the examples in section two and in this section show that the notion of a tree-decomposition is intimately related to that of monadic composition. If, as in section two, we restrict the language $L^k_T$ to predicates that are relevant to the underlying graph, and consider our as our models only the graphs portions of our nostrums we can conclude that the resulting theory of such infinite graphs is decidable.
References


ABSTRACT

THE COMPOSITION METHOD

by

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The subject of this dissertation is a model-theoretic decidability technique for monadic second-order theories. This technique, which we call the composition method, was used by Saharon Shelah and Yuri Gurevich to extend decidability results for linear orders in some sense as far as possible. Many properties of structures that are of importance in computer science are expressible in monadic second-order logic. There is a close connection between this logic and the theory of finite automata and regular languages. Unfortunately, the composition method has remained relatively unknown. Part of the purpose of this dissertation is to present the technique in an accessible manner, and to show that it provides a clarifying and unifying approach to many decision problems. This is done in part by giving new applications of the technique to problems for which other and varied methods have been developed. Two of these applications concern classes of graphs whose tree-width is bounded by some fixed number, and are novel uses of the composition method in the sense that they allow identification of elements in components of the composition. We also give a new general template for the use of the method (which allows for the identification mentioned above) in the form of theorems that give sufficient conditions for its application.
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