1. Evaluate the complex line integral

\[ \int_C z \, dz, \]

where \( C \) is the path \( x = 2t, y = t^2 \), for \(-2 \leq t \leq 2\).

A: We parametrize the path \( C \) as \( z(t) = 2t + it^2 \) for \(-2 \leq t \leq 2\), and so \( z'(t) = 2 + 2it \). Next, on the path \( \overline{z} = 2t - it^2 \). Therefore,

\[
\int_C z \, dz = \int_{-2}^{2} (2t - it^2)(2 + 2it) \, dt = \int_{-2}^{2} (4t + 2t^3 + 2it^2) \, dt
\]

\[
= (2t^2 + \frac{1}{2}t^4 + \frac{2}{3}it^3)|_{-2}^{2} = 16 + \frac{16}{3}i - (16 + \frac{16}{3}i) = \frac{32}{3}i.
\]
2. Assume that the function \( f(z) \) is analytic in a bounded domain \( D \), and \( |f(z)| = m \) in \( D \), where \( m \) is a real constant. What can you say about \( f \)?

A: Then \( f(z) = \text{const.} \). Indeed, if \( m = 0 \) then \( f = 0 \). If \( m \neq 0 \) we write \( f = u + iv \), then \( |f|^2 = u^2 + v^2 = m^2 \). We differentiate with respect to \( x \) and \( y \), divide by 2, and obtain

\[
uu_x + vv_x = 0, \quad uu_y + vv_y = 0.
\]

Next, we use the Cauchy-Riemann equations \( u_x = v_y \) and \( u_y = -v_x \), multiply the first equation by \( u \), the second by \( v \), and find

\[
u^2 u_x - vv u_y = 0, \quad uu_y + v^2 u_x = 0.
\]

Adding both equations yields

\[
0 = (u^2 + v^2)u_x = m^2 u_x \implies u_x = 0.
\]

Similarly, multiplying the first equation by \( v \) and the second by \( u \) leads to

\[
u v u_x + v^2 v_x = uu v_y - v^2 u_y = 0, \quad u^2 u_y + uu v_y = 0.
\]

Subtracting the equations yields

\[
0 = (u^2 + v^2)u_y = m^2 u_y \implies u_y = 0.
\]

Since \( u_x = 0 \) and \( u_y = 0 \) it follows that \( u = \text{const.} \), and then it follows from the Cauchy-Riemann equations that \( v = \text{const.} \), and so \( f = \text{const.} \).

3. Let \( z \) be a complex number, show that

\[
\cos^2 z + \sin^2 z = 1.
\]

A: We have

\[
\cos^2 z = \frac{1}{4}(e^{iz} + e^{-iz})^2 \quad \sin^2 z = -\frac{1}{4}(e^{iz} - e^{-iz})^2.
\]

Therefore,

\[
\cos^2 z + \sin^2 z = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz} - e^{2iz} - 2 + e^{-2iz}) = 1.
\]
4. Find the complex line integral \( \oint_C f(z) \, dz \) of the function

\[
f(z) = z^2 + \frac{1}{z} + \frac{1}{z - 4} + \frac{1}{(z - 4)^2},
\]

where \( C \) is either (i) the circle \(|z| = 1\), or (ii) the circle \(|z| = 12\). (You do not need to compute the integrals, but you must explain carefully which formulas you used, and fully justify your answer).

A: Since \( z^2 \) is analytic everywhere in \( \mathbb{C} \), its contribution to either line integral is zero. \( 1/z \) has a singularity at the origin, and \( \oint_C (1/z) \, dz = 2\pi i \) for either curve \( C \). The functions \( 1/(z-4) \) and \( (z-4)^2 \) are analytic on and inside the first curve. For the second curve we have

\[
\oint_{|z|=12} \frac{1}{z-4} \, dz = 2\pi i, \quad \oint_{|z|=12} \frac{1}{(z-4)^2} \, dz = 0.
\]

Therefore,

\[
\oint_{|z|=1} f(z) \, dz = 2\pi i, \quad \oint_{|z|=12} f(z) \, dz = 4\pi i.
\]

5. Determine where do the following functions satisfy the Cauchy-Riemann equations.

(i) \( f(z) = iz + |z|^2 \),

(ii) \( f(z) = z^2 - iz \),

(iii) \( f(z) = \ln(1/z) \).

A: (i) We have \( f = u + iv = (x^2 + y^2 - y) + i(x) \), and so \( v_y = 0 \) while \( u_x = 2x \), also \( u_y = 2y - 1 \) and \( v_x = 1 \), and the Cauchy-Riemann equations are satisfied only at \( z = 0 \).

(ii) \( f' = 2z - i \), so \( f \) is analytic in \( \mathbb{C} \) and the Cauchy-Riemann equations are satisfied everywhere.

(iii) We have \( \ln(1/z) = \ln 1 - \ln(z) \pm 2n\pi i = -\ln(z) \pm 2n\pi i \). Then,

\[
f' = -\frac{1}{z}.
\]

The function is analytic everywhere except at \( z = 0 \), and so it satisfies the Cauchy-Riemann equations everywhere except at \( z = 0 \).
6. Evaluate the complex line integral
\[ \int_C \frac{2}{z} \, dz, \]
where \( C \) is the straight line segment from \( z = 1 + i \) to \( z = 9 + 9i \).

A: We have that \( f = \frac{2}{z} = (2 \ln(z))' = F' \). Since \( F = 2 \ln(z) \) is analytic in any neighborhood of \( C \), we obtain
\[ \int_C \frac{2}{z} \, dz = F(9 + 9i) - F(1 + i) = 2 \ln \left( \frac{9 + 9i}{1 + i} \right) = 2 \ln 9. \]

7. Find all \( z \) such that \( e^z = 1 + 2i \).

A: Taking the ln of both sides yields
\[ z = \ln(1 + 2i) = \ln(\sqrt{5}) + i \text{Arg}(1 + 2i) \pm 2\pi ni, \]
for \( n = 0, 1, 2, \ldots \). Finally,
\[ \text{Arg}(1 + 2i) = \tan^{-1}(2). \]

8. Write the following expressions as \( a + ib \):
   (i) \( \sin^2(1 + i) \); (ii) \( \sin(e^i) \).

A: (i) We can use either of two ways. We have \( \sin(z) = (\exp(iz) - \exp(-iz))/2i \), thus,
\[ \sin^2(1 + i) = -\frac{1}{4} \left( e^{2i(1+i)} - 2 + e^{-2i(1+i)} \right) = \frac{1}{4} \left( e^{-2+2i} - 2 + e^{2-2i} \right) = \frac{1}{4} \left( 2 - e^{-2}(\cos 2 + i \sin 2) - e^2(\cos 2 - i \sin 2) \right) = \frac{1}{2} (1 - \cos 2 \cosh 2) + \frac{1}{2} i \sin 2 \sinh 2. \]

(ii) We have \( e^i = \cos(1) + i \sin(1) \), thus,
\[ \sin(e^i) = \sin(\cos(1) + i \sin(1)) = \sin(\cos(1)) \cosh(1) + i \cos(\cos(1)) \sinh(1). \]
9. (You have to answer this question) Evaluate the line integral

\[ I = \oint_C \left( \frac{e^z}{(z + 1)^2 + 2z} \right) \, dz, \]

where \( C \) is the unit circle \(|z| = 2\).

A: We split the integral into two parts. Let \( z = 2e^{it} \) for \( 0 \leq t \leq 2\pi \), and then \( z' = 2ie^{it} \), and so

\[
I_1 = 2 \oint_C \bar{z} \, dz = 2 \int_0^{2\pi} 2e^{-it} 2ie^{it} \, dt = 8i \int_0^{2\pi} dt = 16\pi i.
\]

Next, we use Cauchy's Integral Formula for the derivative and obtain

\[
I_2 = \oint_C \frac{e^z}{(z + 1)^2} \, dz = 2\pi i (e^z)' \big|_{z=-1} = 2\pi e^{-1} i.
\]

We conclude that

\[
I = I_1 + I_2 = 16\pi i + 2\pi e^{-1} i = 2\pi i (8 + e^{-1}).
\]

10. (You have to answer this question) Let \( z \) and \( w \) be two nonzero complex numbers. What is the relationship between \( \ln(z/w) \) and \( \ln(z), \ln(w) \)?

A: From the definitions

\[
\ln \left( \frac{z}{w} \right) = \ln \left| \frac{z}{w} \right| + i \arg \left( \frac{z}{w} \right) \pm 2\pi ni
\]

\[
= \ln |z| - \ln |w| + i \text{Arg}(z) - i \text{Arg}(w) \pm 2\pi ni,
\]

for \( n = 0, 1, 2, \ldots \). Next, \n
\[
\ln(z) = \ln |z| + i \text{Arg}(z) \pm 2\pi ki, \quad \ln(w) = \ln |w| + i \text{Arg}(w) \pm 2\pi mi,
\]

for \( k, m = 0, 1, 2, \ldots \).

Therefore, every number \( \hat{z} \) such that \( \hat{z} = \ln(z/w) \) is also such that \( \hat{z} = \ln(z) - \ln(w) \), so the sets on both sides include the same numbers, so they are the same.