THE FIXED POINT PROPERTY FOR SUBSETS OF SOME CLASSICAL BANACH SPACES

P. N. DOWLING, C.J. LENNARD, AND B. TURETT

ABSTRACT. We prove a general result about the stability of the fixed point property in closed bounded convex subsets of certain Banach spaces. This allows us to characterize those closed bounded convex subsets of $L^1[0,1]$ that are weakly compact. Similarly, we characterize the closed bounded convex subsets of $\ell^1$ that are compact. Generalizations of these results to the setting of noncommutative $L^1$-spaces are also obtained.

1. Introduction

In this paper we prove that if a closed bounded convex subset $K$ of a Banach space contains an asymptotically isometric $\ell^1$-basis, then $K$ contains a closed convex subset which fails the fixed point property for nonexpansive mappings (see Theorem ?? for a precise statement). This result can be viewed a local version of a result of the first two authors [?]. We use this result to characterize the closed bounded convex subsets of $L^1[0,1]$ that are weakly compact and to characterize the closed bounded convex subsets of $\ell^1$ that are compact, thereby answering two questions posed by the first author in [?]. Furthermore, this result can be used to characterize the closed bounded convex subsets of $C_1$, the space of trace class operators, which are weakly compact. Further characterizations of weak compactness are obtained in noncommutative $L^1$-spaces.

While the main result depends on a relatively new concept (asymptotically isometric $\ell^1$-basis), the result has it roots in a paper of Goebel and Kuczumow [?]. In [?], an example is constructed of a decreasing sequence of closed convex subsets of $\ell^1$ such that the even-indexed terms have the fixed point property and the odd-indexed terms fail the fixed point property, thereby showing that the fixed point property is very unstable. Our original intent was to construct a non-compact closed bounded convex subset of $\ell^1$ with the property that all its closed bounded convex subsets have the fixed point property. We will see in Corollary ?? that it is impossible to construct such a set in $\ell^1$.

2. The results

Throughout this note we use standard Banach space terminology as is used the text of Diestel [?]. For more information on fixed point theory, we refer the reader to the book of Goebel and Kirk [?].

We begin with our main result:

\[ 1991 \text{ Mathematics Subject Classification. Primary 47H10, 47H09, 46E30.} \]
\[ \text{The first author was supported in part by a Miami University Summer Research Appointment.} \]
Theorem 1. Let $X$ be a Banach space with a norm $\| \cdot \|$, and let $K$ be a closed bounded convex subset of $X$. Let $(\varepsilon_n)$ be a null sequence in $(0,1)$. If $K$ contains a sequence $(x_n)$ such that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n) |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} (1 + \varepsilon_n) |t_n|,$$

for all $(t_n) \in \ell^1$, then $K$ contains a non-empty closed convex subset $C$ such that there is a nonexpansive affine mapping $T : C \to C$ which fails to have a fixed point in $C$.

Remark. The sequence $(x_n)$ in Theorem 1 is called an asymptotically isometric $\ell^1$-basis. This notion was developed by the first two authors in [?] and was used to show that the nonreflexive subspaces of $L^1[0,1]$ fail the fixed point property. The notion was also noted by J. Hagler [?] as an equivalent condition for the dual of a real Banach space to contain a subspace isometric to $L^1[0,1]$. Until recently [?], Hagler’s result was unpublished and went unnoticed.

Proof of Theorem 1. By passing to subsequences if necessary, we can assume that for each $m \in \mathbb{N}$,

$$(1 + \varepsilon_{2m+1})(3^{-2m-2} + 2^{-1}) + (1 + \varepsilon_{2m+2})(3^{-2m-2})\quad < (1 - \varepsilon_{2m-1})(3^{-2m} + 2^{-1}) + (1 - \varepsilon_{2m})(3^{-2m}). \quad (1)$$

For each $n \in \mathbb{N}$, define

$$\beta_j^{(n)} = \begin{cases} 3^{-j} & \text{if } j \neq 2n - 1, 2n \\ 3^{-2n+1} + 3^{-2n} + 2^{-1} & \text{if } j = 2n - 1 \\ 0 & \text{if } j = 2n \end{cases}$$

Note that $\beta_j^{(n)} \geq 0$ for each $j, n \in \mathbb{N}$ and $\sum_{j=1}^{\infty} \beta_j^{(n)} = 1$ for all $n \in \mathbb{N}$. Hence since $K$ is closed, bounded and convex, $y_n = \sum_{j=1}^{\infty} \beta_j^{(n)} x_j \in K$ for all $n \in \mathbb{N}$.

Define $q = \sum_{j=1}^{\infty} 3^{-j} x_j$ and note that for each $n \in \mathbb{N}$, $y_n = q + u_n$, where $u_n = (3^{-2n} + 2^{-1}) x_{2n-1} - 3^{-2n} x_{2n}$.

Define $C = \overline{\co \{ y_n : n \in \mathbb{N} \}}$. Note that $C$ is a closed convex subset of $K$ and, since $(u_n)$ is equivalent to the unit vector basis of $\ell^1$, we have

$$C = q + \overline{\co \{ u_n : n \in \mathbb{N} \}}$$

$$= q + \left\{ \sum_{n=1}^{\infty} t_n u_n : t_n \geq 0 \text{ for all } n \in \mathbb{N}, \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\}$$

$$= \left\{ \sum_{n=1}^{\infty} t_n y_n : t_n \geq 0 \text{ for all } n \in \mathbb{N}, \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\}.$$
\[ \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1, \] be elements of \( C \) with \( z \neq w \). Then

\[
\|z - w\| = \left\| \sum_{n=1}^{\infty} t_n y_n - \sum_{n=1}^{\infty} s_n y_n \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} t_n u_n - \sum_{n=1}^{\infty} s_n u_n \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} (t_n - s_n) u_n \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} (t_n - s_n) \left[ (3^{-2n} + 2^{-1}) x_{2n-1} - 3^{-2n} x_{2n} \right] \right\|
\]

\[
\geq \sum_{n=1}^{\infty} |t_n - s_n| \left[ (1 - \varepsilon_{2n-1})(3^{-2n} + 2^{-1}) + (1 - \varepsilon_{2n})3^{-2n} \right].
\]

On the other hand, we also have

\[
\|Tz - Tw\| = \left\| \sum_{n=1}^{\infty} t_n y_{n+1} - \sum_{n=1}^{\infty} s_n y_{n+1} \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} t_n u_{n+1} - \sum_{n=1}^{\infty} s_n u_{n+1} \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} (t_n - s_n) u_{n+1} \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} (t_n - s_n) \left[ (3^{-2n-2} + 2^{-1}) x_{2n+1} - 3^{-2n-2} x_{2n+2} \right] \right\|
\]

\[
\leq \sum_{n=1}^{\infty} |t_n - s_n| \left[ (1 + \varepsilon_{2n+1})(3^{-2n-2} + 2^{-1}) + (1 + \varepsilon_{2n+2})3^{-2n-2} \right].
\]

Combining these two inequalities and using inequality (??), along with the fact that since \( x \neq y, t_n \neq s_n \) for some \( n \in \mathbb{N} \), we easily see that \( \|Tx - Ty\| < \|x - y\| \). This completes the proof. \( \square \)

In the proof of the main result of [?], the first author shows that if \( K \) is a closed bounded convex subset of \( L^1[0, 1] \) which is not weakly compact, then \( K \) contains a sequence such that a translate of this sequence by a certain fixed element of \( L^1[0, 1] \) is an asymptotically isometric \( \ell^1 \)-basis. Combining this with Theorem ?? we obtain the following result, which answers Question 3 of [?].

**Corollary 2.** If \( K \) is a closed bounded convex subset of \( L^1[0, 1] \) which is not weakly compact, then \( K \) contains a closed convex non-empty subset \( C \) such that there is a nonexpansive affine mapping \( T : C \to C \) which fails to have a fixed point in \( C \).

This corollary can now be used to characterize weak compactness of closed bounded convex subsets of \( L^1[0, 1] \).

**Corollary 3.** Let \( K \) be a closed bounded convex subset of \( L^1[0, 1] \). Then the following are equivalent;

\[
\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1, \]
(a) $K$ is weakly compact,
(b) Every closed convex subset of $K$ has the fixed point property for continuous affine self-maps,
(c) Every closed convex subset of $K$ has the fixed point property for nonexpansive affine self-maps.

Proof. It is obvious that (b) implies (c). Also (c) implies (a) follows from Corollary ???. Lastly, the implication (a) implies (b) follows from a result of D.P. Milman and V.D. Milman [?, Section 4].

Remark. It is important to note that the word affine cannot be dropped from Corollary ?? because of Alspach’s example of a closed bounded convex subset of $L^1[0,1]$ which is weakly compact but fails the fixed point property for nonexpansive mappings (see [?]). However, in the case of subsets of $\ell^1$ we have no such difficulties and we can easily get the following result which answers Question 8 of [?].

**Corollary 4.** A closed bounded convex subset $K$ of $\ell^1$ is compact if and only if every closed convex subset of $K$ has the fixed point property for nonexpansive self-maps.

The results in Corollaries ??, ?? and ?? have noncommutative analogues that follow from the work in [?]. The main result of [?] is that if $\mathcal{M}$ is a semi-finite von Neumann algebra equipped with a faithful, normal, semi-finite trace, then every nonreflexive subspace of the predual of $\mathcal{M}$, $\mathcal{M}_*$, contains an asymptotically isometric copy of $\ell^1$. However, upon close analysis of the proof, one sees that if $K$ is a closed bounded convex subset of $\mathcal{M}_*$ which is not weakly compact, then $K$ contains a sequence such that a translate of this sequence by a certain fixed element of $\mathcal{M}_*$ is an asymptotically isometric $\ell^1$-basis. Therefore, following the same strategy as in Corollary ??, we obtain the following two results which are analogous to Corollaries ?? and ??.

**Corollary 5.** If $K$ is a closed bounded convex subset of $\mathcal{M}_*$ which is not weakly compact, then $K$ contains a closed convex non-empty subset $C$ for which there is a nonexpansive affine mapping $T : C \to C$ which fails to have a fixed point in $C$.

**Corollary 6.** Let $K$ be a closed bounded convex subset of $\mathcal{M}_*$. Then the following are equivalent;

(a) $K$ is weakly compact,
(b) Every closed convex subset of $K$ has the fixed point property for continuous affine self-maps,
(c) Every closed convex subset of $K$ has the fixed point property for nonexpansive affine self-maps.

The noncommutative analogue of $\ell^1$ is $\mathcal{C}_1$, the space of trace class operators. The second author proved that $\mathcal{C}_1$ has the weak fixed point property [?]. Using this result along with Corollary ?? we get the noncommutative analogue of Corollary ??.

**Corollary 7.** A closed bounded convex subset $K$ of $\mathcal{C}_1$, the space of trace class operators, is weakly compact if and only if every closed convex subset of $K$ has the fixed point property for nonexpansive self-maps.
THE  FIXED  POINT  PROPERTY  FOR  SUBSETS

References

3. S.J. Dilworth, Maria Girardi and J. Hagler, Dual Banach spaces which contain an isometric copy of $L_1$, preprint

DEPARTMENT OF MATHEMATICS AND STATISTICS, MIAMI UNIVERSITY, OXFORD, OH 45056
E-mail address: doulinpn@muohio.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA 15260
E-mail address: lennard+pitt.edu
Current address: Department of Mathematics and Statistics, Miami University, Oxford, OH 45056
E-mail address: lennarcj@muohio.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, OAKLAND UNIVERSITY, ROCHESTER, MI 48309
E-mail address: turett@veia.acs.oakland.edu